WALSH FUNCTIONS AND HADAMARD MATRICES

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Abstract. The aim of this work is to summarize the basic properties of the three main systems consisting of Walsh functions in the theory of Fourier analysis. These systems are the Walsh-Paley system, the original Walsh system and the Walsh-Kaczmarz system. We study the relations between them focused on the construction on Hadamard matrices. Finally, we give some formulae which make it possible the fast calculation of Dirichlet and Fejér kernels based on these systems.

1. Systems consisting of Walsh functions

In the classical Fourier theory the so-called trigonometric system is used for approximation. Instead of trigonometric functions, the dyadic harmonic analysis uses functions which take only values 1 and -1 on the interval $[0,1[$. For instance, consider the function

$$r(x) := \begin{cases} 1 & x \in [0, \frac{1}{2}[\\ -1 & x \in [\frac{1}{2}, 1[\end{cases}$$

and extend it to the real numbers $\mathbf{R}$ by periodicity 1. The extended function r allows us to introduce the concept of Rademacher system given by the functions

$$r_k(x) := r(2^k x) \qquad (k \in \mathbf{N},\ x \in [0,1[),$$

where $\mathbf{N}$ is the set of non-negative integers.

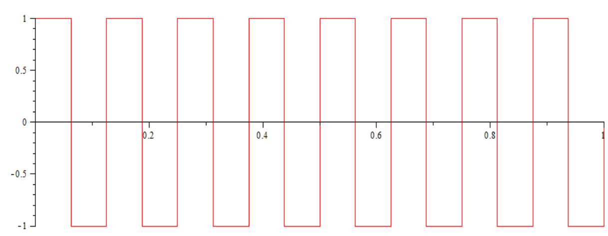

Figure 1. The Rademacher function r_3

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Rademacher functions take alternatively the values 1 and -1, moreover r_k is constant on the dyadic intervals

$$I_{k+1}(i) := \left[\frac{i-1}{2^{k+1}}, \frac{i}{2^{k+1}}\right[\qquad (i = 1, \dots, 2^{k+1})$$

and

$$r_k(x) = \operatorname{sgn}(\sin(2^{k+1}\pi x)) \qquad (x \in [0,1[)$$

with the exception of dyadic numbers of the form $\frac{i}{2^{k+1}}$, where $0 \le i < 2^{k+1}$.

The fact that the Rademacher function r_k takes exactly 2^k times the value 1 and 2^k times the values -1 on intervals with the same measure implies that

$$\int_0^1 r_k(x)\,dx = 0$$

and for the same reason, the integral of r_k is also zero on the sets $I_{l+1}(i)$ where $l < k$ and $1 \le i \le 2^{l+1}$. Thus,

$$\begin{aligned} \int_0^1 r_k(x) r_l(x)\,dx &= \sum_{i=1}^{2^{l+1}} \int_{I_{l+1}(i)} r_k(x) r_l(x)\,dx \\ &= \sum_{i=1}^{2^{l+1}} r_l\left(\frac{i-1}{2^{l+1}}\right) \int_{I_{l+1}(i)} r_k(x)\,dx = 0. \end{aligned}$$

Since $r_k^2 \equiv 1$ for all $k \in \mathbf{N}$, the Rademacher system is orthonormal on $L^2([0,1[)$.

However, we can prove similarly the fact that the integral on $[0,1[$ of the product of arbitrary finite many Rademacher functions is also zero, from which it follows that the Rademacher system is not complete on $L^2([0,1[)$. Indeed, the function $r_0 r_1$ is orthogonal to any Rademacher function. We call Walsh function the product of finite many Rademacher functions, but it was not the original definition. A system consisting of all Walsh functions is an orthonormal and complete system on $L^2([0,1[)$.

The original Walsh system was introduced by Walsh [7] in 1923. His definition was recursive and probably he did not known the Rademacher system introduced a year before. In order to write the original Walsh system as the product of Rademacher functions we introduce the following notation.

Every $n \in \mathbf{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k = 0$ or $n_k = 1$ for all $k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the dyadic expansion of n. Similarly the dyadic expansion $(x_0, x_1, \dots)$ of a real number $x \in [0,1[$ is given by the sum

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}},$$

where $x_k = 0$ or $x_k = 1$ for all $k \in \mathbf{N}$. This expansion is not unique if x is a dyadic rational. When this situation occurs we choose the expansion terminates in zeros. That is, the expansion with an index l such that $x_k = 0$ for all $k > l$. By the dyadic expansion Rademacher functions can be expressed as follows

$$r_k(x) = (-1)^{x_k} \qquad (x \in [0,1[,\ k \in \mathbf{N}).$$

The original Walsh system ϕ can be written by Rademacher functions as follows

$$\phi_n(x) = \prod_{k=0}^{\infty} r_k^{n_k+n_{k+1}}(x) \qquad (x \in [0,1[, n \in \mathbf{N})$$

Paley [3] was first to recognize that Walsh functions are products of Rademacher functions. In 1932 he introduced the system ω called Walsh-Paley system by

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[, n \in \mathbf{N}).$$

In order to find a connection between the arrangement of the original Walsh system an the Walsh-Paley system we define the dyadic sum of a pair of non-negative integers n and m by

$$n \oplus m = \sum_{k=0}^{\infty} |n_k - m_k| 2^k,$$

where $(n_0, n_1, \dots)$ and $(m_0, m_1, \dots)$ are de dyadic expansion of the integers n and m respectively. Since the integer quotient $\left[\frac{n}{2}\right]$ has the dyadic expansion $(n_1, n_2, \dots)$, we obtain immediately from the definition of the systems ϕ and ω the relation

$$\phi_n(x) = \omega_{n \oplus \left[\frac{n}{2}\right]}(x) \qquad (x \in [0,1[, n \in \mathbf{N}). \tag{1}$$

Conversely, denote by $\bigoplus_{k=0}^{\infty} \left[\frac{n}{2^k}\right]$ the dyadic sum of all of the integer quotients $\left[\frac{n}{2^k}\right]$, where $k = 0, 1, \dots$. Thus, the dyadic expansion of $\bigoplus_{k=0}^{\infty} \left[\frac{n}{2^k}\right]$ is

$$(n_0 + n_1 + n_2 + \dots, n_1 + n_2 + n_3 + \dots, n_2 + n_3 + n_4 + \dots, \dots),$$

from which we obtain the relation

$$\omega_n(x) = \phi_{\bigoplus_{k=0}^{\infty} \left[\frac{n}{2^k}\right]}(x) \qquad (x \in [0,1[, n \in \mathbf{N}). \tag{2}$$

The original Walsh system satisfies the requirements for $\psi\alpha$ systems defined by Gát [1] in 1991. It means the fact that the original Walsh system can be written as $\phi_n = \omega_n \alpha_n$, where the functions α_n are composed of the product of functions α_k^j $(k, j \in \mathbf{N})$ defined on the interval $[0,1[$ with the following properties:

- for every $k \in \mathbf{N}$ the functions α_k^j are constant on the dyadic intervals $I_k(i)$ $(i = 1, 2, \dots, 2^k)$ for all $j \in \mathbf{N}$.
- $|\alpha_k^j| = 1$ and $\alpha_0^j = \alpha_k^0 = \alpha_k^j(0) = 1$ for all $k, j \in \mathbf{N}$.
- by the notation $n^{(k)} = \sum_{i=k}^{\infty} n_i 2^i$ define the functions α_n as the product

$$\alpha_n(x) := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}(x) \qquad (x \in [0,1[).$$

Indeed, let $\alpha_k^j = r_{k-1}^{j_k}$ for all positive integer k and for all $j \in \mathbf{N}$ with dyadic expansion $(j_0, j_1, \dots)$. Thus, the functions α_k^j have the properties above and

$$\alpha_n(x) = \prod_{k=0}^{\infty} r_k^{n_{k+1}}(x) \qquad (x \in [0,1[)$$

from which we obtain the relation $\phi_n = \omega_n \alpha_n$ for all $n \in \mathbf{N}$.

The third system studied in this paper is the so-called Walsh-Kaczmarz system κ introduced by Šneider [6] in 1948.

$$\kappa_0(x) := 1, \quad \kappa_n(x) := r_A(x) \prod_{k=0}^{A-1} r_k^{n_{A-k-1}}(x) \qquad (n \in \mathbf{N}, x \in [0,1[).$$

where $2^A \leq n < 2^{A+1}$. For any $k \in \mathbf{N}$ define the map $\tau_k \colon \mathbf{N} \to \mathbf{N}$ by the formula

$$\tau_k(n) := \sum_{i=0}^{k-1} n_{k-i-1} 2^i + \sum_{i=k}^{\infty} n_i 2^i.$$

Note that τ_k reverses the first k bites of the dyadic expansion of every non-negative integer, i.e. the dyadic expansion of $\tau_k(n)$ is

$$(n_{k-1}, n_{k-1}, \ldots, n_1, n_0, n_k, n_{k+1}, \ldots),$$

from which we have $\kappa_0(x) = \omega_0(x)$ and

$$\kappa_n(x) = \omega_{\tau_k(n)}(x), \qquad \omega_n(x) = \kappa_{\tau_k(n)}(x) \qquad (x \in [0,1[) \tag{3}$$

for all positive integer n such that $2^k \leq n \leq 2^{k+1}$.

Similarly we can also define τ_k as a bit-reversing transformation on the dyadic expansion of any $x \in [0,1[$, that is $\tau_k \colon [0,1[\to [0,1[$,

$$\tau_k(x) := \sum_{i=0}^{k-1} \frac{x_{k-i-1}}{2^{i+1}} + \sum_{i=k}^{\infty} \frac{x_i}{2^{i+1}} \qquad (k \in \mathbf{N}).$$

In this case τ_k is a measure-preserving transformation such that $\tau_k(\tau_k(x)) = x$ for all $k \in \mathbf{N}$ and $x \in [0,1[$. Moreover,

$$\kappa_{2^k+m}(x) = r_k(x)\omega_{\tau_k(m)}(x) = r_k(x)\omega_m(\tau_k(x)) \qquad (x \in [0,1[) \tag{4}$$

where $0 \leq m < 2^k$.

2. Hadamard matrices

Hadamard matrices are square matrices with orthogonal rows whose entries are either 1 or -1. Due to the fact that systems ϕ, ω and κ consist of orthogonal functions with values 1 and -1 which are piecewise-constant on the dyadic intervals $I_k(j)$ $(j = 1, 2, \ldots, 2^k)$ for all indexes $n < 2^k$, matrices $\mathcal{O}^{(k)}$, $\mathcal{W}^{(k)}$ and $\mathcal{K}^{(k)}$ with entries

$$\mathcal{O}_{i,j}^{(k)} := \phi_{i-1}\left(\frac{j-1}{2^k}\right), \quad \mathcal{W}_{i,j}^{(k)} := \omega_{i-1}\left(\frac{j-1}{2^k}\right), \quad \mathcal{K}_{i,j}^{(k)} := \kappa_{i-1}\left(\frac{j-1}{2^k}\right)$$

$(i, j = 1, 2, \ldots, 2^k)$ are Hadamard matrices of type $2^k \times 2^k$ for all $k \in \mathbf{N}$. Thus, we call matrices $\mathcal{O}$, $\mathcal{W}$ and $\mathcal{K}$ Hadamard-Walsh, Hadamard-Paley and Hadamard-Kaczmarz matrices respectively.

For a fixed $k \in \mathbf{N}$ matrices $\mathcal{O}^{(k)}$, $\mathcal{W}^{(k)}$ and $\mathcal{K}^{(k)}$ have the same rows in different order. Relations (1), (2) and (3) give us the way to rearrange Hadamard matrices into each others. Another way to construct Hadamard matrices is based on iteration. In order to simplify the notations define the Kronecker product of the matrices A and B by

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,m}B \\ \vdots & \vdots & & \vdots \\ a_{n,1}B & a_{n,2}B & \dots & a_{m,m}B \end{pmatrix}$$

where $a_{i,j}$ ($i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$) are the entries of the matrix A.

First, we deal with Hadamard-Paley matrices. By the definition of the Walsh-Paley system it is not difficult to prove that

$$\omega_n = r_k \omega_{n-2^k} \qquad (2^k \leq n < 2^{k+1})$$

from which we have

$$\mathcal{W}^{(k+1)} = \begin{pmatrix} \mathcal{W}^{(k)} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \mathcal{W}^{(k)} \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{pmatrix} \qquad (k \in \mathbf{N}).$$

In other words, the first half of $\mathcal{W}^{(k+1)}$ is obtained duplicating all elements of $\mathcal{W}^{(k)}$ and the second half is similar, but we have to change the sign of every second elements. We can see this fact in the following example.

$$\mathcal{W}^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\mathcal{W}^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

In case of the original Walsh system, the first half of $\mathcal{O}^{(k+1)}$ is also obtained duplicating all of elements of $\mathcal{O}^{(k)}$. By the definition of ϕ we can prove that

$$\phi_n = r_k r_{k-1} \phi_{n-2^k} \qquad (2^k \leq n < 2^{k+1}).$$

The values of the piecewise-constant function $r_k r_{k-1}$ on the dyadic intervals $I_{k+1}(j)$ ($j = 1, 2, \dots, 2^{k+1}$) form a consecutive sequence of numbers $1, -1, -1, 1$ repeated 2^{k-1} times. For this reason the second half of $\mathcal{O}^{(k+1)}$ is also similar to the first half, but here we have to change the signs of the elements according with the sequence mentioned before, so in every segment of four element the sign of the second and third elements must be changed. We can see this fact in the following example.

$$\mathcal{O}^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\mathcal{O}^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

However, the situation is quite different for Hadamard-Kaczmarz matrices, because we can not write an adequate relation for κ_n and κ_{n-2^k}. However, by (4) we have

$$\kappa_n = r_k \omega_{\tau_k(n-2^k)} \qquad (2^k \leq n < 2^{k+1}). \tag{5}$$

The relation above gives us the opportunity to obtain the second half of the matrix $\mathcal{K}^{(k+1)}$ from the Kronecker product of the bit-reversal rearrangement of the matrix $\mathcal{W}^{(k)}$ and the matrix $(1 \;\; -1)$, but this method, at first glance, it does not seem appropriate because it needs several calculations. The introduction of Hadamard matrices $\mathcal{H}$ solves this problem. These matrices are defined recursively as

$$\mathcal{H}^{(0)} := (1), \qquad \mathcal{H}^{(1)} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathcal{H}^{(k+1)} := \mathcal{H}^{(1)} \otimes \mathcal{H}^{(k)},$$

for every positive integer k. Moreover, for any $i = 0, 1, \dots, 2^k - 1$ denote by $h_i^{(k)}$ the piecewise-constant function on the dyadic intervals $I_k(j)$ $(j = 1, 2, \dots, 2^k)$ such that

$$h_i^{(k)}(\frac{j}{2^k}) = \mathcal{H}_{i+1,j}^{(k)}.$$

By the Kronecker product the matrix

$$\mathcal{H}^{(k)} = \begin{pmatrix} \mathcal{H}^{(k-1)} & \mathcal{H}^{(k-1)} \\ \mathcal{H}^{(k-1)} & -\mathcal{H}^{(k-1)} \end{pmatrix}$$

is partitioned into four blocks. Fix a $0 \leq i < 2^k$ and $x \in [0, 1[$ with expansion $(i_0, i_1, \dots)$ and $(x_0, x_1, \dots)$ respectively. The binary coefficients i_{k-1} and x_0 determine the block in which the value of $h_i^{(k)}(x)$ appears. In particular

$$h_i^{(k)}(x) = -1^{i_{k-1} x_0} h_{i - i_{k-1} 2^{k-1}}^{(k-1)}(x - \frac{x_0}{2}) = r_0^{i_{k-1}}(x) h_{i - i_{k-1} 2^{k-1}}^{(k-1)}(x - \frac{x_0}{2}),$$

from which we obtain by iteration the formula

$$h_i^{(k)}(x) = r_0^{i_{k-1}}(x) r_1^{i_{k-2}}(x) \dots r_{k-1}^{i_0}(x) = \omega_{\tau_k(i)}(x) \qquad (x \in [0, 1[). \tag{6}$$

Hence we can rewrite (5) as follows

$$\kappa_n = r_k h_{n-2^k}^{(k)} \qquad (2^k \leq n < 2^{k+1}) \tag{7}$$

to obtain

$$\mathcal{K}^{(k+1)} = \begin{pmatrix} \mathcal{K}^{(k)} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \mathcal{H}^{(k)} \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{pmatrix} \qquad (k \in \mathbf{N}).$$

We can see this fact in the following example.

$$\mathcal{K}^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad \mathcal{H}^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\mathcal{K}^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

The Kronecker product is implemented in most Computer Algebra Systems which allows the a fast calculation of Hadamard matrices $\mathcal{H}$. For this reason, it is especially interesting how to obtain Hadamard matrices $\mathcal{W}^{(k)}$, $\mathcal{O}^{(k)}$ and $\mathcal{K}^{(k)}$ as the row rearrangement of Hadamard matrices $\mathcal{H}^{(k)}$.

By (6) and the property $\tau_k(\tau_k(i)) = i$ we have

$$\omega_i = h^{(k)}_{\tau_k(i)} \qquad (i = 0, 1, \ldots, 2^k - 1). \tag{8}$$

Thus, we obtain Hadamard-Paley matrices from the relation

$$\mathcal{W}^{(k)}_i = \mathcal{H}^{(k)}_{\tau_k(i-1)+1} \qquad (i = 1, \ldots, 2^k).$$

By (1) and (6) we have

$$\phi_i = \omega_{i\oplus\left[\frac{i}{2}\right]} = h^{(k)}_{\tau_k(i\oplus\left[\frac{i}{2}\right])} \qquad (i = 0, 1, \ldots, 2^k - 1). \tag{9}$$

Thus, we obtain Hadamard-Walsh matrices from the relation

$$\mathcal{O}^{(k)}_i = \mathcal{H}^{(k)}_{\tau_k((i-1)\oplus\left[\frac{i-1}{2}\right])+1} \qquad (i = 1, \ldots, 2^k).$$

By (3) and (6) we have

$$\kappa_0 = h^{(k)}_0, \qquad \kappa_i = \omega_{\tau_A(i)} = h^{(k)}_{\tau_k(\tau_A(i))} \qquad (i = 1, \ldots, 2^k - 1), \tag{10}$$

where A is the range of the positive integer i, i.e. $A := \max\{k \in \mathbf{N} \colon i_k = 1\}$. Since $A < k$, it is not difficult to see that the number $\tau_k(\tau_A(i))$ has the dyadic expansion

$$(\underbrace{0, 0, \ldots, 0}_{k-A-1}, 1, \underbrace{i_0, i_1, \ldots, i_{A-1}}_{A}),$$

hence $\tau_k(\tau_A(i)) = (2(i - 2^A) + 1)2^{k-A-1}$. Thus, we obtain Hadamard-Kaczmarz matrices from the relation

$$\mathcal{K}^{(k)}_1 = \mathcal{H}^{(k)}_1, \qquad \mathcal{K}^{(k)}_{i+1} = \mathcal{H}^{(k)}_{(2(i-2^A)+1)2^{k-A-1}} \qquad (i = 1, \ldots, 2^k - 1).$$

3. Dirichlet kernels

Dirichlet kernels are the finite sums of system functions. In particular denote

$$D^{\psi}_n(x) := \sum_{i=0}^{n-1} \psi_i(x) \qquad (x \in [0, 1[,\ n \in \mathbf{N}), \tag{11}$$

where ψ represents one of the system ϕ, ω or κ. Obviously, in most cases the kernels D^{ϕ}_n, D^{ω}_n and D^{κ}_n are different functions, but they are equal if $n = 2^k$ $(k \in \mathbf{N})$, since for ϕ, ω or κ system functions with index less than 2^k are the same, but with different enumeration (see Section 1). Denote this common Dirichlet kernel by D_{2^k}. The values of these kernels are very simple (see [4]).

Lemma 1 (Paley's lemma)**.**

$$D_{2^k} = \begin{cases} 2^k & x \in I_k(1) \\ 0 & x \in [0,1[\setminus I_k(1) \end{cases}$$

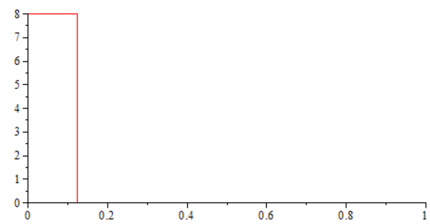

FIGURE 2. The Dirichlet kernel D_8

Paley's lemma can be proved by iteration as follows. For any positive integer k we have

$$D_{2^k} - D_{2^{k-1}} = \sum_{i=2^{k-1}}^{2^k-1} \omega_i = \sum_{i=0}^{2^{k-1}-1} \omega_{2^{k-1}+i} = \sum_{i=0}^{2^{k-1}-1} r_{k-1}\omega_i = r_{k-1} D_{2^{k-1}}.$$

Thus,

$$D_{2^k}(x) = (1 + r_{k-1}(x)) D_{2^{k-1}}(x) \qquad (x \in [0,1[). \tag{12}$$

By the dyadic expansion $(x_0, x_1, \dots)$ of x

- if $x_{k-1} = 0$ then $r_{k-1}(x) = 1$, so $D_{2^k}(x) = 2D_{2^{k-1}}(x)$,
- if $x_{k-1} = 1$ then $r_{k-1}(x) = -1$, so $D_{2^k}(x) = 0$.

Since $D_{2^0} = \omega_0 = 1$, $D_{2^k}(x)$ is not zero if and only if $x_0 = x_1 = \cdots = x_{k-1} = 0$ and then $D_{2^k}(x) = 2^k$ from which the statement of Paley's lemma holds.

Paley's lemma allows us to obtain a fast iteration for Dirichlet kernels in case of systems ω and ϕ. Let $n = 2^k + m$ where $0 \le m < 2^k$. Thus,

$$D_n^{\omega} = \sum_{i=0}^{2^k-1} \omega_i + \sum_{i=2^k}^{n-1} \omega_i = D_{2^k} + \sum_{j=0}^{m-1} \omega_{2^k+j} = D_{2^k} + \sum_{j=0}^{m-1} r_k \omega_j$$

from which we have

$$D_n^{\omega} = D_{2^k} + r_k D_m^{\omega}. \tag{13}$$

Similarly we can prove

$$D_n^{\phi} = D_{2^k} + r_k r_{k-1} D_m^{\phi}. \tag{14}$$

It is not possible to obtain a similar iteration formula for the Walsh-Kaczmarz system. Notwithstanding, the relation (7) allows us to find another way

$$D_n^{\kappa} = D_{2^k} + \sum_{j=0}^{m-1} \kappa_{2^k+j} = D_{2^k} + r_k \sum_{j=0}^{m-1} h_j^{(k)}. \tag{15}$$

In other words, we obtain a fast calculation from Dirichlet kernels D^{κ} from the sums of the rows of Hadamard matrix $\mathcal{H}^{(k)}$ instead $\mathcal{K}^{(k+1)}$ which has four times more entries.

We remark that equations (13), (14) and (15) are also valid for $m = 2^k$.

It is possible to obtain Dirichlet kernels based on different systems from each other. In Section 1 we show that ϕ is a $\psi\alpha$ system where the functions α_n are given by

$$\alpha_n(x) = \prod_{k=0}^{\infty} r_k^{n_{k+1}}(x) \qquad (x \in [0,1[).$$

In [2] Gát gave the connexion between Dirichlet kernels concerning $\psi\alpha$ systems and the Walsh system (or a Vilenkin system in a more generalized form). Through this connexion

$$D_n^{\phi}(x) = \alpha_n(x) D_n^{\omega}(x) \qquad (x \in [0,1[) \tag{16}$$

holds. Figure 3 shows the Dirichlet kernels D_{27}^{ω} and D_{27}^{ϕ} plotted in the same graphic to illustrate (16). We shift down a little bit the graph of D_{27}^{ϕ} to avoid the superposition of the lines.

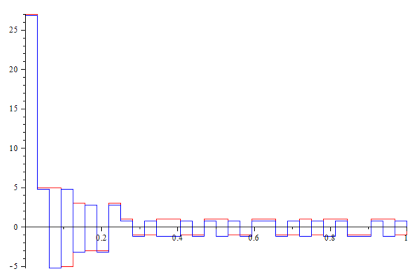

Figure 3. Dirichlet kernels with index 27 for the Walsh-Paley (red) and the original Walsh (blue) system

The approach for Dirichlet kernels of the Walsh-Kaczmarz system is based on the transformation $\tau_k(x)$ where $x \in [0,1[$. By (6) and (15) we have

$$D_n^{\kappa}(x) = D_{2^k}(x) + r_k(x) \sum_{j=0}^{m-1} \omega_j(\tau_k(x)) \qquad (x \in [0,1[).$$

Since the functions $D_{2^k}(x)$ and $r_k(x)$ do not depend on the permutation of the first k coefficient of the dyadic expansion of x, we obtain $D_{2^k}(x) = D_{2^k}(\tau_k(x))$ and $r_k(x) = r_k(\tau_k(x))$, hence

$$D_n^{\kappa}(x) = D_n^{\omega}(\tau_k(x)) \qquad (x \in [0,1[). \tag{17}$$

Figure 4 shows the Dirichlet kernels D_{44}^{ω} and D_{44}^{κ} plotted in the same graphic to illustrate (17). We shift down a little bit the graph of D_{44}^{κ} to avoid the superposition of the lines.

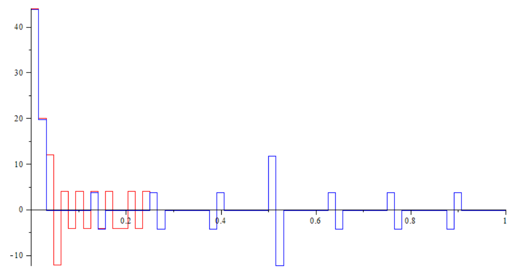

Figure 4. Dirichlet kernels with index 44 for the Walsh-Paley (red) and the Walsh-Kaczmarz (blue) system

Finally, we deal with the Lebesgue constants of the systems above, i.e. the L^1-norm of Dirichlet kernels. In this regard denote

$$L_n^{\psi} := \int_0^1 |D_n^{\psi}(x)|\,dx \qquad (n \in \mathbf{N}).$$

where ψ represents one of the system ϕ, ω or κ. By (16) we obtain

$$L_n^{\phi} := \int_0^1 |D_n^{\phi}(x)|\,dx = \int_0^1 |\alpha_n(x) D_n^{\omega}(x)|\,dx = \int_0^1 |D_n^{\omega}(x)|\,dx = L_n^{\omega},$$

since $|\alpha_n(x)| = 1$ for all $n \in \mathbf{N}$ and $x \in [0,1[$. On the other hand, by (17) and from the fact that the transformation τ_k is measure-preserving we obtain

$$L_n^{\kappa} := \int_0^1 |D_n^{\kappa}(x)|\,dx = \int_0^1 |D_n^{\omega}(\tau_k(x))|\,dx = \int_0^1 |D_n^{\omega}(x)|\,dx = L_n^{\omega},$$

for all $n \in \mathbf{N}$ and $x \in [0,1[$, where k is given by the relation $2^k \le n < 2^{k+1}$.

In summary, Lebesgue constants of the systems with which we deal in this work are the same. For his reason, we use the same notation L_n for any of three cases. Moreover, L_n can be obtained recursively (see [4]) as follows

$$L_{2^k+m} = 1 + L_m - \frac{m}{2^k}$$

for all $k \in \mathbf{N}$ and $2^k \le m < 2^{k+1}$.

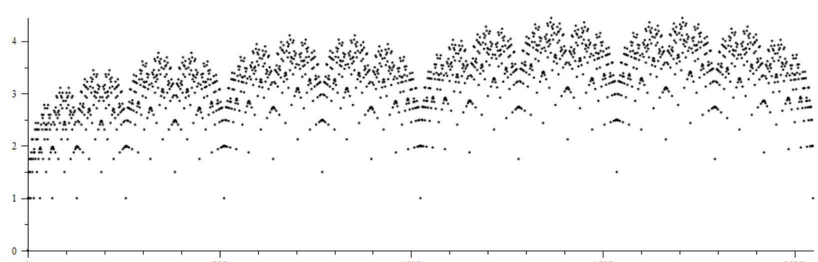

Figure 5. The Lebesgue constants

4. Fejér kernels

Fejér kernels are the average of Dirichlet kernels. In particular define

$$K_n^{\psi}(x) := \frac{1}{n}\sum_{i=1}^{n} D_n^{\psi}(x) \qquad (x \in [0,1[,\ n \in \mathbf{P}, K_0^{\psi} := 0),$$

where ψ represents one of the system ϕ, ω or κ and $\mathbf{P}$ the set of all positive integers. Fejér kernels can be also obtained directly from the system functions as follows

$$K_n^{\psi}(x) = \sum_{i=0}^{n-1}\left(1 - \frac{i}{n}\right)\psi_i(x) \qquad (x \in [0,1[,\ n \in \mathbf{P}).$$

First we focus our attention on the study of Fejér kernels with indices which are the powers of 2. By (13) for the Walsh-Paley system (see [4]) we have

$$\begin{aligned} 2^k K_{2^k}^{\omega} - 2^{k-1} K_{2^{k-1}}^{\omega} &= \sum_{i=2^{k-1}+1}^{2^k} D_i^{\omega} = \sum_{i=1}^{2^{k-1}} D_{2^{k-1}+i}^{\omega} = \sum_{i=1}^{2^{k-1}} (D_{2^{k-1}} + r_{k-1} D_i^{\omega}) \\ &= 2^{k-1} D_{2^{k-1}} + r_{k-1}\sum_{i=1}^{2^{k-1}} D_i^{\omega} = 2^{k-1} D_{2^{k-1}} + r_{k-1} 2^{k-1} K_{2^{k-1}}^{\omega}. \end{aligned}$$

Thus,

$$2^k K_{2^k}^{\omega}(x) = 2^{k-1} D_{2^{k-1}}(x) + (1 + r_{k-1}(x)) 2^{k-1} K_{2^{k-1}}^{\omega}(x) \qquad (x \in [0,1[). \tag{18}$$

By the dyadic expansion $(x_0, x_1, \dots)$ of x

- suppose $x_{k-1} = 0$. Then $r_{k-1}(x) = 1$, so $2^k K_{2^k}^{\omega}(x) = 2^{k-1} D_{2^{k-1}}(x) + 2^k K_{2^{k-1}}^{\omega}(x)$. Hence, by Paley's lemma, if there exists an index $j < k-1$ such that $x_j = 1$ then
$$K_{2^k}^{\omega}(x) = K_{2^{k-1}}^{\omega}(x),$$
but if $x_0 = x_1 = \cdots = x_{k-2} = 0$ then by (4) and the fact that $D_i^{\omega}(0) = i$ for all $i \in \mathbf{N}$, we obtain
$$K_{2^k}^{\omega}(x) = \frac{1}{2^k}\sum_{i=1}^{2^k} i = \frac{2^k + 1}{2}.$$
- suppose $x_{k-1} = 1$. Then $r_{k-1}(x) = -1$, so $2^k K_{2^k}^{\omega}(x) = 2^{k-1} D_{2^{k-1}}(x)$. Hence, by Paley's lemma, if there exists another index $j < k-1$ such that $x_j = 1$ then
$$K_{2^k}^{\omega}(x) = 0,$$
but if $x_0 = x_1 = \cdots = x_{k-2} = 0$ then
$$K_{2^k}^{\omega}(x) = 2^{k-2}.$$

Thus, by iteration it is not difficult to see that $K_{2^k}^{\omega}(x)$ is not zero if and only if $x_0 = x_1 = \cdots = x_{k-1} = 0$, i.e. $x \in I_k(1)$, or there is only one index $j < k$ such that $x_j = 1$, i.e $x \in I_k(2^{k-j-1} + 1)$. In particular

$$K_{2^k}^{\omega}(x) = \begin{cases} \frac{2^k+1}{2} & x \in I_k(1) \\ 2^{k-j-2} & x \in I_k(2^j + 1),\ j = 0, 1, \dots, k-1 \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

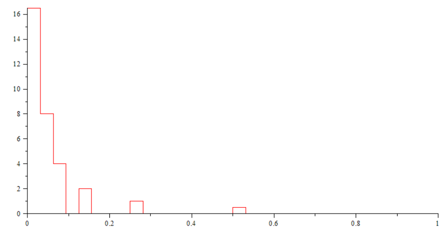

FIGURE 6. The Fejér kernel K_{32}^{ω}

By (14) for the original Walsh system we have

$$\begin{aligned}2^kK_{2^k}^{\phi}-2^{k-1}K_{2^{k-1}}^{\phi}&=\sum_{i=2^{k-1}+1}^{2^k}D_i^{\phi}=\sum_{i=1}^{2^{k-1}}D_{2^{k-1}+i}^{\phi}=\sum_{i=1}^{2^{k-1}}(D_{2^{k-1}}+r_{k-1}r_{k-2}D_i^{\phi})\\&=2^{k-1}D_{2^{k-1}}+r_{k-1}r_{k-2}\sum_{i=1}^{2^{k-1}}D_i^{\phi}=2^{k-1}D_{2^{k-1}}+r_{k-1}r_{k-2}2^{k-1}K_{2^{k-1}}^{\phi}.\end{aligned}$$

Thus,

(20)
$$2^kK_{2^k}^{\phi}(x)=2^{k-1}D_{2^{k-1}}(x)+(1+r_{k-1}(x)r_{k-2}(x))2^{k-1}K_{2^{k-1}}^{\phi}(x)\qquad(x\in[0,1[).$$

By the dyadic expansion $(x_0,x_1,\dots)$ of x

- suppose $x_{k-1}=0$.
  - if $x_{k-2}=1$ then $r_{k-1}(x)r_{k-2}(x)=-1$ and $D_{2^{k-1}}(x)=0$, hence

(21)
$$K_{2^k}^{\phi}(x)=0.$$

  - if $x_{k-2}=0$ then $r_{k-1}(x)r_{k-2}(x)=1$, so $2^kK_{2^k}^{\phi}(x)=2^{k-1}D_{2^{k-1}}(x)+2^kK_{2^{k-1}}^{\phi}(x)$. Hence, by Paley's lemma, if there exists an index $j<k-2$ such that $x_j=1$ we have $K_{2^k}^{\phi}(x)=K_{2^{k-1}}^{\phi}(x)$. By iteration, if $j:=\max\{i<k-2\colon x_i=1\}$ then
$$K_{2^k}^{\phi}(x)=K_{2^{k-1}}^{\phi}(x)=\cdots=K_{2^{j+2}}^{\phi}(x)=0,$$
since $K_{2^{j+2}}^{\phi}(x)$ satisfies the conditions of (21). On the other hand, similarly to the Walsh-Paley system, if $x_0=x_1=\cdots=x_{k-3}=0$ then
$$K_{2^k}^{\phi}(x)=\frac{2^k+1}{2}.$$
- suppose $x_{k-1}=1$.
  - if $x_{k-2}=1$ then $r_{k-1}(x)r_{k-2}(x)=1$ and $D_{2^{k-1}}(x)=0$, hence
$$K_{2^k}^{\phi}(x)=K_{2^{k-1}}^{\phi}(x).$$
  - if $x_{k-2}=0$ then $r_{k-1}(x)r_{k-2}(x)=-1$, so $2^kK_{2^k}^{\phi}(x)=2^{k-1}D_{2^{k-1}}(x)$. Hence, by Paley's lemma, if there exists an index $j<k-2$ such that $x_j=1$ then
$$K_{2^k}^{\phi}(x)=0,$$
but if $x_0=x_1=\cdots=x_{k-3}=0$ then
$$K_{2^k}^{\phi}(x)=2^{k-2}.$$

Thus, by iteration it is not difficult to see that $K^{\phi}_{2^k}(x)$ is not zero if and only if $x_0 = x_1 = \cdots = x_{k-1} = 0$, i.e $x \in I_k(1)$, or there is an index $j < k$ such that $x_0 = x_1 = \cdots = x_{j-1} = 0$ and $x_j = x_{j+1} = \cdots = x_{k-1} = 1$, i.e. $x \in I_k(2^{k-j})$. In particular

$$K^{\phi}_{2^k}(x) = \begin{cases} \frac{2^k+1}{2} & x \in I_k(1) \\ 2^{k-j-1} & x \in I_k(2^j),\ j = 1, 2, \ldots, k \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

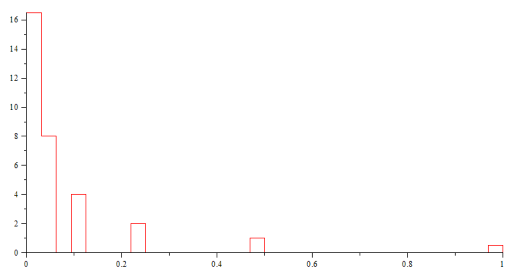

FIGURE 7. The Fejér kernel K^{ϕ}_{32}

Finally, by (15) for the Walsh-Kaczmarz system (see [5]) we have

$$2^j K^{\kappa}_{2^j}(x) - 2^{j-1} K^{\kappa}_{2^{j-1}}(x) = \sum_{i=2^{j-1}+1}^{2^j} D^{\kappa}_i(x) = \sum_{i=1}^{2^{j-1}} D^{\kappa}_{2^{j-1}+i}(x) \tag{23}$$

$$= \sum_{i=1}^{2^{j-1}} (D_{2^{j-1}}(x) + r_{j-1}(x) D^{\omega}_i(\tau_{j-1}(x))) \tag{24}$$

$$= 2^{j-1} D_{2^{j-1}}(x) + r_{j-1}(x) \sum_{i=1}^{2^{j-1}} D^{\omega}_i(\tau_{j-1}(x)) \tag{25}$$

$$= 2^{j-1} D_{2^{j-1}}(x) + 2^{j-1} r_{j-1}(x) K^{\omega}_{2^{j-1}}(\tau_{j-1}(x)) \tag{26}$$

for all $x \in [0, 1[$. Moreover, $2^0 K^{\kappa}_{2^0} = 1$. Thus, by the sum of the equations above for j from 0 to $k - 1$ we obtain

$$K^{\kappa}_{2^k}(x) = \frac{1}{2^k} + \sum_{j=0}^{k-1} 2^{j-k} D_{2^j}(x) + 2^{j-k} r_j(x) K^{\omega}_{2^j}(\tau_j(x)) \qquad (x \in [0, 1[). \tag{27}$$

Equation (27) means that the construction of Fejér kernels with index 2^k for the Walsh-Kaczmarz system differs from that used for the Walsh-Paley and the original Walsh systems. In the last two cases, we obtain Fejér kernels with index 2^k directly from functions which take frequently the value zero, that is from (19) and (22). For the Walsh-Kaczmarz system we have to sum functions with this property. Indeed, if $x \in I_j(2^i+1)$ then $\tau_j(x) \in I_j(2^{j-i-1}+1)$, hence by (19) we have

$$K_{2^j}^{\omega}(\tau_j(x)) = \begin{cases} \frac{2^j+1}{2} & x \in I_j(1) \\ 2^{i-1} & x \in I_j(2^i+1),\ i = 0, 1, \dots, j-1 \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

for all $j \in \mathbf{N}$. Thus,

$$D_{2^j}(x) + r_j(x) K_{2^j}^{\omega}(\tau_j(x)) = \begin{cases} \frac{3\cdot 2^j+1}{2} & x \in I_{j+1}(1) \\ \frac{2^j-1}{2} & x \in I_{j+1}(2) \\ 2^{i-1} & x \in I_{j+1}(2^{i+1}+1),\ i = 0, 1, \dots, j-1 \\ -2^{i-1} & x \in I_{j+1}(2^{i+1}+2),\ i = 0, 1, \dots, j-1 \\ 0 & \text{otherwise.} \end{cases}$$

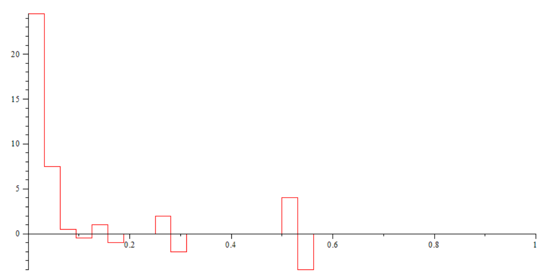

FIGURE 8. $D_{2^4}(x) + r_4(x) K_{2^4}^{\omega}(\tau_4(x))$

We discuss now Fejér kernels with general $n = 2^k + m$ indexes. We start by the decomposition

$$nK_n^{\psi} = \sum_{i=1}^{2^k} D_i^{\psi} + \sum_{i=2^k+1}^{n} D_i^{\psi} = 2^k K_{2^k}^{\psi} + \sum_{i=1}^{m} D_{2^k+i}^{\psi}$$

where ψ is one of the system ϕ, ω or κ. Thus by (13) we have

$$nK_n^{\omega} = 2^k K_{2^k}^{\omega} + mD_{2^k} + r_k m K_m^{\omega}, \tag{29}$$

and by (14) we have

$$nK_n^{\phi} = 2^k K_{2^k}^{\phi} + mD_{2^k} + r_k r_{k-1} m K_m^{\phi}, \tag{30}$$

which give us a useful recursive formulae to obtain Fejér kernels based on the Walsh-Paley and the original Walsh system.

On the other hand, by (15) we have

$$nK_n^{\kappa} = 2^k K_{2^k}^{\kappa} + mD_{2^k} + r_k \sum_{i=0}^{m-1} (m-i) h_i^{(k)}, \tag{31}$$

which gives us a useful formula in to obtain Fejér kernels based on the Walsh-Kaczmarz system from Hadamard matrices.

We remark the formulae (29), (30) and (30) are also valid for $m = 2^k$. Comparing the formula (30) for $m = 2^k$ with (23) for $j = k + 1$ we have

$$\sum_{i=0}^{2^k-1} (2^k - i) h_i^{(k)}(x) = 2^k K_{2^k}^{\omega}(\tau_k(x)) \qquad (k \in \mathbf{N},\ x \in [0,1[).$$

Since

$$\sum_{i=0}^{2^k-1} (2^k - i) h_i^{(k)} = 2^k \sum_{i=0}^{2^k-1} h_i^{(k)} - \sum_{i=0}^{2^k-1} i h_i^{(k)} = 2^k D_{2^k} - \sum_{i=0}^{2^k-1} i h_i^{(k)}$$

we have by (28) the formula

$$\begin{aligned}\sum_{i=0}^{2^k-1} i h_i^{(k)}(x) &= 2^k D_{2^k}(x) - 2^k K_{2^k}^{\omega}(\tau_k(x)) \\ &= \begin{cases} 2^{2k-1} - 2^{k-1} & x \in I_k(1) \\ -2^{k+i-1} & x \in I_k(2^i + 1),\ i = 0, 1, \ldots, k-1 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

which gives us an interesting formula with respect to Hadamard matrices $\mathcal{H}^{(k)}$.

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