Convergence in norm on the complete product of finite groups

Doktori (PhD) értekezés

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Ezen értekezést a Debreceni Egyetem TTK matematika és számítás-tudományok Doktori Iskola matematikai analízis, függvényegyenletek programja keretében készítettem a Debreceni Egyetem TTK doktori (PhD) fokozatának elnyerése céljából.

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a jelölt aláírása

Tanúsítom, hogy Dr. Toledo Rodolfo doktorjelölt 2002 – 2003 között a fent megnevezett Doktori Iskola matematikai analízis, függvényegyenletek programja keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javasolom.

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a témavezető aláírása

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Preface

The theory of abstract harmonic analysis has been a relevant progress in the last decades. An increasing number of mathematicians have adopted the point of view that the most appropriate setting for the development of the theory of Fourier analysis is furnished by the class of all locally compact groups. Starting of the classical theory of Fourier series and integrals the relative ease with which the basic concepts and theorems can be transferred to this general context in the abelian case is not valid for the non-commutative case. For instance, it is well known that the Riemann-Lebesgue lemma is not valid for non-commutative cases.

The structure of topological groups was extensively studied in the years 1925-1940, and the subject is far from dead even today. The study of the direct products of topological groups have been started since the beginning of the theory of topological groups. Pontryagin [20] examined very extensively the structure of countable direct products treated special cases of finite direct products. Vilenkin [1] obtained several results for the commutative cases.

The dyadic group is the simplest but nontrivial model of the complete product of finite groups. Representing the characters of the dyadic group ordered in the Paley's sense, we obtain the Walsh system. This system is applied in the processing of data but has an interesting theoretical point of view.

A natural generalization on the Walsh-Paley system is the Vilenkin system introduced by Vilenkin [36] in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case. In Hungary a dyadic analysis team works leaded by Schipp having several results in this theory. For instance, they proved that the Paley theorem is true for an arbitrary Vilenkin group, i.e. the partial sums of the Vilenkin-Fourier series of a function in $L^p(G)$ $(1 < p < \infty)$ converge in the appropriate norm to the function (Young [39], Schipp [21], Simon [26]).

The example above is not true for all cases if we take the complete product of arbitrary finite group (not necessarily commutative). The study of this groups is the aim of this work. These studies were appeared in [10] by Gát and Toledo first and they obtained not only negative results for this groups, because they also proved the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case.

This work is organized as follows. The first chapter is introductory, introducing the topology, the measure and the system with which we work. This kind of system is called (by the author) a representative product system because we use representation theory to collect the functions appeared in it. The Weyl-Peter's theorem ensure the orthonormality and completeness in L^2 of this system. Representing this system on the interval $[0, 1]$ we plot and show relevant examples. In this chapter we use the notation appeared in [13] and [14].

Chapter 2 summarizes the results of [10]. We introduce the basic concepts of Fourier analysis and give the properties of the Dirichlet kernels to study the convergence in norm of Fourier series and Fejér means. We also introduce the concept of modulus of continuity to give class of functions for which the partial sums of it's Fourier series converge to the function in L^1 or in the uniform norm. Finally, we obtain an important positive result, i.e. if G is a bounded group, the Fejér means of a function $f \in L^p(G)$, $1 \leq p \leq \infty$ converge to the function in L^p-norm.

In Chapter 3 we estimate the Fourier coefficients which not necessarily tend to zero, using the modulus of continuity of the function and the uniform norm of the system. We specially study the functions with bounded fluctuation. On the other hand we consider an interesting class of functions, namely the ones that are constant on every conjugacy classes. The system of characters of the representations is complete in this class of functions, so we use characters to study the absolute convergence of series constructed in this way.

Chapter 4 treats the general case of product system adapting the results of Schipp [21] for the convergence in Hardy and BMO norms. We use the convergence of operators with property Δ to study the conjugate martingale transforms defined on not necessarily bounded Vilenkin group. In this chapter the notation of the different Hardy and BMO spaces is the same to the notation appeared in [38].

Finally, the author would like to thank Professor Dr. F. Schipp for his valuable ideas and Professor Dr. G. Gát for carefully reading this work and for his several advices and remarks to improve this work.

Chapter 1

The structure of the complete product of finite groups

In Section 1.1 we resume the elementary topological properties of compact totally disconnected groups and specially the topology defined on the complete product of finite groups (Proposition 1.1.1). Pontryagin [20] studied heavily the structure of these groups. An ample resume of the characteristics of compact totally disconnected groups appears in [13, Chapter II]. In this section we also introduce the Haar measure with which we work. More about Haar measure appear in [13, Chapter IV].

In Section 1.2 we state some facts about the representation theory of compact, not necessarily abelian groups, closing this section with the famous Weyl-Peter's theorem 1.2.3. (see [14]). The statements in Section 1.2 justify the notations in Section 1.3, where we give the structure of the groups and introduce the concept of representative product systems. Vilenkin [1] investigated very extensively the commutative cases. Some relevant results for product systems was obtained by Schipp in [21] and [22]. Approximation questions for the non-abelian cases were studied first by Gát-Toledo in [10].

Various examples of representative product systems appear in Section 1.4 and 1.5 representing them on the interval $[0,1]$ using Fine's map. The Theorems 1.5.2 and 1.5.1 are proved by Morgenthaler [15] for the Walsh group and also appear in the book [24]. Toledo resume these results in [35] using the analogy with Vilenkin groups to show the appearance of the studied systems on the interval $[0,1[$ and utilize the MAPLE software to compute that.

Throughout this work denote by $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ the set of non-negative integers, real and complex numbers, respectively. Denote by $|A|$ the cardinal number of the set A. In order to simplicity we always use the multiplication to denote the group operation and use the symbol e to denote the identity of the groups. The notation which we used in this chapter is similar to the one appeared in [13].

1.1 Facts about topology and measure

A *topological group* G is an entity which is both a group and a topological space and the group operations and the topology are appropriately connected, namely the mappings $(x,y) \to xy$ of $G \times G$ onto G and $x \to x^{-1}$ of G onto G are continuous. The algebraic properties of the group affect the topological properties of the space and vice versa. For example, a topological group which satisfies the separation axiom T_0 is completely regular. In addition, an open basis $\mathfrak{U}$ at the group identity e gives an open base for G by the family $\{xU : x \in G,\ U \in \mathfrak{U})$ and similarly by the family $\{Ux : x \in G,\ U \in \mathfrak{U})$.

The subgroup H of a topological group G is also a topological group with its relative topology as a subspace of G. Define a topology of G/H by the following rule: the set $\overline{X} := \{xH : x \in X\}$ is open in G/H if an only if the set $\bigcup_{x\in X} xH$ is open in G. Then G/H is a discrete topological space if an only if the H is open in G. If G is compact the space, then G/H is also compact, so if H is open and compact, then G/H is a finite set with discrete topology. Let N be a normal subgroup of G, then G/N is a topological group.

A subset of a topological space is *connected* if it is not the disjoint union of two nonvoid open sets. A topological space is *totally disconnected* if all of its components are points. Component is a connected subset which is properly contained in no other connected subset. A topological space is *0-dimensional* if the family of all open and closet sets is an open basis for the topology.

Through this work (see 1.3) denote by G the complete direct product of the finite groups G_k $(k \in \mathbf{N})$. Assume that all of groups G_k have the discrete topology and the topology of G is the topological product of the discrete topologies. Thus

Proposition 1.1.1. *G is an infinite compact totally disconnect group. Hence*

(a) G is a 0-dimensional topological group.

(b) the topology of G is metrizable.

(c) G has a countable open basis $G = I_0 \supset I_1 \supset \ldots$ at the identity e consisting of open and closed normal compact totally disconnect subgroups, where e is the intersection of this subgroups.

Let $\mathcal{A}$ be the smallest σ-algebra of subsets of G which contain all open subsets of G. The members of $\mathcal{A}$ are called the *Borel sets* of G. A measure ν defined on G is said to be *regular* if for every open set U, we have $\nu(U) = \sup\{\nu(F) : F \text{ is compact and } F \subseteq U\}$; for all $A \in \mathcal{A}$, we have $\nu(A) = \inf\{\nu(U) : U \text{ is open and } U \supseteq A\}$. If $\nu(xA) = \nu(Ax) = \nu(A)$ for all $x \in G$ and $A \in \mathcal{A}$, then ν is said to be *two-sided invariant*. Since G is compact there is an unique non-negative regular measure μ of the Borel sets of G which is two-sided translation invariant and $\mu(G) = 1$. This measure is called the *normalized Haar measure* of G.

Proposition 1.1.2. *Denote by $m_k := |G_k|$ the cardinal number of the finite group G_k ($k \in \mathbf{N}$) and μ_k the corresponding normalized Haar measure. Moreover, denoted by G the complete direct product of G_k ($k \in \mathbf{N}$) with normalized Haar measure μ. Then*

(a) for every set $A \in G_k$ and function $f : G_k \to \mathbf{C}$

$$\mu_k(A) = \frac{|A|}{m_k} \qquad \text{and} \qquad \int_{G_k} f \, d\mu_k = \frac{1}{m_k} \sum_{x \in G_k} f(x).$$

(b) since the normalized Haar measure is unique, μ is the product measure of μ_k's ($k \in \mathbf{N}$).

1.2 Representative functions

A *representation* U of a group G is a homomorphism of G into the semigroup of all operators defined in some linear space E over an arbitrary field F. That is, $U : x \to U_x$ such that $U_x : E \to E$ is a linear transformation for all $x \in G$ and

$$U_{xy} = U_x U_y \qquad (x, y \in G).$$

The linear space E is called the *representation* space of U, and let the dimension of a representation be the *dimension* of its own representation space.

We can assume that U_e is the identity operator on E, because E is the direct sum of invariant subspaces E_0 and E_1 such that $U_x(E_0) = \{0\}$ for all $x \in G$, and U_e is the identity operator on E_1, hence we can take E by E_1.

Throughout this work suppose that the representation space of all representations is a reflexive complex Banach space which is a topological linear space under the metric and norm induced by the inner product $\langle .,. \rangle$. The representation U is called *unitary* if all of operators U_x are unitary, i.e. U_x is a linear isometry of E onto E. A representation U with representation space E is called *irreducible* if only the spaces $\{0\}$ and E are invariant under all operators U_x $(x \in G)$.

We can define an equivalence relation in the set of all continuous irreducible unitary representations of the group G in the following manner. Two representations U and U' with representation spaces E and E' respectively are equivalent if there is a bounded linear isometry $T : E \to E'$ such that

$$U'_x T = T U_x \qquad (x \in G).$$

Denote by Σ the set of all equivalence classes induced by the above relation. Σ is called the *dual object* (Σ) of the group G. The common dimension of all representations in the class $\sigma \in \Sigma$ is denoted by d_σ. All group have a trivial representation with dimension 1, namely the one which is identically equal to 1. A representation with dimension 1 is called a *character*, i.e. a character is a mapping $\chi : G \to \mathbf{C}$ such that

$$\chi(xy) = \chi(x)\chi(y) \quad (x, y \in G), \qquad |\chi(x)| = 1 \quad (x \in G).$$

Proposition 1.2.1. *Let G be a finite group. Then*

(a) $|\Sigma|$ is equal to the number of conjugacy class in G. (The system of the conjugacy classes is a partition of G induced by the equivalence relation: $a \sim b$ if and only if $\exists x \in G : \; a = xbx^{-1}$).

(b) if $\Sigma = \{\sigma_1, \sigma_1 \dots \sigma_{|\Sigma|}\}$, then $|G| = d_{\sigma_1}^2 + d_{\sigma_2}^2 + \cdots + d_{\sigma_{|\Sigma|}}^2$.

(c) d_{σ_i} is a divisor of $|G|$ $(1 \le i \le |\Sigma|)$.

(d) if the group G is abelian, then $|\Sigma| = |G|$ and all representations of G are characters.

(e) if the group G is not abelian, then there is a representation with dimension greater than 1.

Proposition 1.2.2. *Let G be the complete direct product of the finite groups G_k ($k \in \mathbf{N}$). Then*

(a) since G is compact, the set Σ is countable and the dimensions of all representations of G are finite.

(b) U is a continuous irreducible representation of G if and only if U is the tensor product of finite numbers of continuous irreducible representations of distinct groups G_k.

Let $U^{(\sigma)}$ be a continuous irreducible representation in the class σ of the dual object of a compact group. Functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \qquad i, j \in \{1, \dots, d_\sigma\}$$

are called *coordinate functions* for $U^{(\sigma)}$, where $\xi_1, \dots, \xi_{d_\sigma}$ is an orthonormal basis in the representation space of $U^{(\sigma)}$. Coordinate functions play an important role in the description of the structure of $L^2(G) := \{f : G \to \mathbf{C} : \|f\|^2 := \int_G |f|^2 \, d\mu < \infty\}$, i.e.:

Theorem 1.2.3 (Weyl-Peter). *Let G be a compact group. Then for all $\sigma \in \Sigma$ and $j, k \in \{1, 2, \dots, d_\sigma\}$ the set of functions $\sqrt{d_\sigma} u_{j,k}^{(\sigma)}$ is an orthonormal basis for $L^2(G)$. Thus for $f \in L^2(G)$, we have*

$$f = \sum_{\sigma \in \Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma \hat{f}(i, j, \sigma) u_{j,k}^{(\sigma)}, \tag{1.1}$$

where

$$\hat{f}(i, j, \sigma) := \int_G f \overline{u_{j,k}^{(\sigma)}} \, d\mu$$

and the series in (1.1) converges in the metric of $L^2(G)$. Furthermore, if $\{a_{j,k}^{(\sigma)} : j,k \in \{1,2,\ldots,d_\sigma; \sigma \in \Sigma\}$ is any set of complex numbers such that

$$\sum_{\sigma\in\Sigma}\sum_{j,k=1}^{d_\sigma} d_\sigma |a_{j,k}^{(\sigma)}|^2 < \infty,$$

there is a unique function g in $L^2(G)$ such that $\hat{f}(i,j,\sigma) = a_{j,k}^{(\sigma)}$ for all $j,k \in \{1,2,\ldots,d_\sigma\}$; $\sigma \in \Sigma\}$ and for which accordingly

$$g = \sum_{\sigma\in\Sigma}\sum_{j,k=1}^{d_\sigma} d_\sigma a_{j,k}^{(\sigma)} u_{j,k}^{(\sigma)}.$$

The finite linear combination of arbitrary coordinate functions are called *representative functions*. Using the $\|f\|_u := \sup\{|f(x)| : x \in G\}$ uniform norm, we have:

Proposition 1.2.4. *Let G be a compact group. Then the set of all representative functions is dense in the set of all continuous function on G with respect to the uniform norm.*

Finally, by Propositions 1.2.2 and b notice that all coordinate functions of the complete direct product of finite groups G_k ($k \in \mathbf{N}$) are the finite product of coordinate functions of distinct groups G_k. An interesting ordering of coordinate functions is given in the following section.

1.3 Representative product systems

Let $m := (m_k,\ k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k, ($k \in \mathbf{N}$). Suppose that each group has discrete topology and normalized Haar measure μ_k. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, \dots)$, where $x_k \in G_k$, ($k \in \mathbf{N}$). We call this sequence the *expansion* of x. The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

Define G^0 as the set of sequences of G terminating in e's (i.e. the set of "finite" sequences), $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \le k < n\} \quad (x \in G, n \in \mathbf{N})$$

$I_n := I_n(e)$. We say that every set $I_n(x)$ is an *interval*. The intervals I_n are a countable neighborhood base at the identity of the product topology on G.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \le n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the expansion of n with respect to m. In this case let $n^* = (n_0, n_1, \dots) \in G$. We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and $n_{(k)} := \sum_{j=0}^{k-1} n_k M_k$, $n^{(k)} = \sum_{j=k}^{\infty} n_k M_k$.

Now we denote by Σ_k the dual object of G_k. Let $\{\varphi_k^s : 0 \le s < m_k\}$ be the set of all *normalized coordinate functions* of the group G_k and suppose that $\varphi_k^0 \equiv 1$. Thus for every $0 \le s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's

Theorem 1.2.3 secure that the system ψ is orthonormal and complete in $L^2(G_m)$.

The functions ψ_n $(n \in \mathbf{N})$ are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N}).$$

Ψ_k is the multiplication of the greatest product of the square root of the dimension and the L^1-norm of the functions φ appeared in all group G_j for $0 \leq j < k$. It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

1.4 Examples

1.4.1 The Walsh system

Let $m_k = 2$ for all $k \in \mathbf{N}$ and $\mathcal{Z}_2$ be the cyclic group of order 2. Thus $G_k = \mathcal{Z}_2$. The characters of $\mathcal{Z}_2$ are the *Rademacher functions*:

$$\varphi_k^s(x) = (-1)^{sx} \qquad (s \in \{0, 1\},\ x \in \mathcal{Z}_2).$$

The product system of φ is called the *Walsh system*. It is easy to see that in this case $\Psi_k \equiv 1$.

1.4.2 Vilenkin systems

Let the sequence m be an arbitrary sequence of integers greater than 1 and $\mathcal{Z}_n$ be the cyclic group of order n, where n is an integer greater than 1. Let $G_k = \mathcal{Z}_{m_k}$ for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_{m_k}$ are the *generalized Rademacher functions*:

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k) \qquad (s \in \{0, \ldots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1).$$

The product system of φ is called a *Vilenkin system*. We also obtain that $\Psi_k \equiv 1$.

1.4.3 The complete product of $\mathcal{S}_3$

Let $m_k = 6$ for all $k \in \mathbf{N}$ and $\mathcal{S}_3$ be the *symmetric group* on 3 elements. Let $G_k = \mathcal{S}_3$ for all $k \in \mathbf{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation ($6 = 1^2 + 1^2 + 2^2$). Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the functions φ_k^s. In the notation the index k is omitted because all of the groups G_k are the same.

	e	(12)	(13)	(23)	(123)	(132)	$\Vert\varphi^s\Vert_1$	$\Vert\varphi^s\Vert_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

φ^2,...,φ^5 correspond to the 2-dimensional representation. Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems. On the other hand, $\max\{\Vert\varphi^s\Vert_1\Vert\varphi^s\Vert_\infty : 0 \le s < 6\} = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$ if $k \to \infty$

1.4.4 The complete product of $\mathcal{Q}_2$

Let $m_k = 8$ for all $k \in \mathbf{N}$ and $\mathcal{Q}_2$ be the the *quaternion group* of order 8, i.e.

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}.$$

Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbf{N}$. $\mathcal{Q}_2$ has four characters and a 2-dimensional representation ($8 = 1^2+1^2+1^2+1^2+2^2$). Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the functions φ_k^s. In the notation the index k is omitted because all of the groups G_k are the same.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$\varphi^4,\ldots,\varphi^7$ correspond to the 2-dimensional representation. Notice that values of $|\varphi^s|$ are 0 or the square of the corresponding dimension only. Hence, the absolute value of the coordinate functions are 0 or 1 respectively. A representation of this form is called a *monomial representation*. If all of the representations are monomial, then $\Psi_k = 1$ for $k \in \mathbf{N}$, but the group G is not necessarily abelian.

1.5 Relation with the interval [0,1]

From 1.1.1 we have that the topology of G is metrizable. Moreover, the metric we concerned is induced by a norm as follows. Order the elements of all groups G_k, $(k \in \mathbf{N})$ in some way such that the first is always their identity. In fact, the ordering is a bijection between G_k and $\{0, 1, \ldots, m_k - 1\}$ which gives to every $x \in G_k$ the integer $0 \le \overline{x} < m_k$ $(\overline{e} = 0)$. Define

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x \in G).$$

It is easy to see that $|.|$ is a norm and the proceeded metric $d(x, y) := |xy^{-1}|$ induces the topology of G. In addition, $0 \leq |x| \leq 1$ for all $x \in G$. Using this fact we represent the group G in the interval $[0, 1]$.

Any $x \in [0, 1]$ can be written

$$x := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (0 \leq \overline{x_k} \leq m_k - 1),$$

but there are numbers with two expressions of this form. They are all numbers in the set

$$\mathbf{Q} := \left\{ \frac{p}{M_n} : 0 \leq p < M_n,\ n, p \in \mathbf{N} \right\}$$

called *m-adic rational numbers* (Note that 1 is not an m-adic rational number). The other numbers have only one expression. The m-adic rational numbers have an expression terminates in 0's and other terminates in $m_k - 1$'s. We choose the first one to make an unique relation for all numbers in the interval $[0, 1]$ with their expression, named the *m-adic expansion* of the number. In this manner we assign to a number in the interval $[0, 1]$ having an m-adic expansion $(\overline{x_0}, \overline{x_1}, \dots)$ an element of G with expansion $(x_0, x_1, \dots)$. We denote this relation by ρ. ρ is called *Fine's map*. Fine's map is an injective map satisfying:

$$\begin{array}{lll} \rho(x+) = \rho(x-) = \rho(x) & & (x \in (0,1) \setminus \mathbf{Q}) \\ \rho(x+) = \rho(x), & \rho(x-) = \rho^*(x) & (x \in \mathbf{Q}) \\ \rho(0+) = \rho(0) = (e, e, \dots), & \rho(1-) = \rho(1) & \end{array}$$

where $\rho^*(x)$ signifies the element of G terminates in $m_k - 1$'s with norm x. $\rho(x+)$ and $\rho(x-)$ signify respectively the right and left limit of ρ at x under the usual metric.

Using Fine's map we introduce a new operation in the interval $[0, 1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x, y \in [0, 1[).$$

We shall remark that the interval $[0,1[$ is not a group under the new operation since it is not associative, but commutative and has identity.

An *m-adic interval* always mean an interval of the form

$$I(n,p) := \left[\frac{p}{M_n}, \frac{p+1}{M_n}\right[\qquad (0 \le p < M_n,\ n,p \in \mathbf{N}).$$

We name the *m-adic topology* the one induced by the m-adic intervals on $[0,1[$. This topology is totally disconnected because the m-adic intervals are both open and closed and form a countable basis. The m-adic topology is issued by the metric:

$$d(x,y) := |\rho(x)\rho(y)^{-1}| \qquad (x,y \in [0,1[).$$

Fine's map give a natural relation between the new structure of $[0,1[$ and the structure of G. ρ is a continuous map under the m-adic topology since for any $x \in G$ and $n \in \mathbf{N}$ we have $\rho^{-1}(I_n(x)) = I(n,p)$, where $p := M_n \sum_{k=0}^{n-1} \frac{\overline{x_k}}{M_{k+1}}$, but this property is not true for the norm $|.|$. In addition

$$\text{(1.2)} \qquad |\rho(x)| = x \qquad (x \in [0,1]),$$

$$\text{(1.3)} \qquad \rho(|x|) = x \ \text{ a.e.} \qquad (x \in G).$$

(1.3) is not true only for the elements of G with expansion terminates in $m_k - 1$'s. From similar reason Fine's map is not a homomorphism but the

$$\text{(1.4)} \qquad \rho(x \odot y) = \rho(|\rho(x)\rho(y)|) = \rho(x)\rho(y)$$

equality is true for all of elements $x, y \in [0,1[$ such that $x \odot y$ is not a m-adic rational.

Let $L^0(G)$ denote the set of all measurable functions on G which are a.e. finite. In some way denote by L^0 the set of all Lebesgue measurable functions on $[0,1]$ which are a.e. finite. According to the Paley lemma (see (2.1.2)) the set of all representative functions on G coincide with the set of all finite linear combinations of characteristics function of intervals, so a function in $L^0(G)$ is a.e. the limit of representative functions.

The following theorem show the relation between the Haar integration on G and the Lebesgue integration on the interval $[0,1]$.

Theorem 1.5.1. *Let ρ denote the Fine's map.*

(a) If $f \in L^0(G)$ then $f \circ \rho \in L^0$. Conversely, if $g \in L^0$ and

$$f(x) := g(|x|) \qquad (x \in G) \tag{1.5}$$

then $f \in L^0(G)$.

(b) If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f \, d\mu = \int_0^1 (f \circ \rho)(x) \, dx.$$

Conversely, if g is Lebesgue integrable and f is defined by (1.5) then f is integrable on G and

$$\int_0^1 g(x) \, dx = \int_G f \, d\mu.$$

Proof. We can prove our statements for characteristics functions of intervals using (1.2) and (1.3). Indeed, if $x \in G$, $n \in \mathbf{N}$ and f is the characteristics function of the interval $I_n(x)$ then g is the characteristics function of the interval $I(n,p)$, where $p := M_n \sum_{k=0}^{n-1} \frac{\overline{x_k}}{M_{k+1}}$. The conversion of the above statement is valid a.e. Then, we obtain our statements for finite linear combinations of characteristics functions of intervals, and finally for the corresponding set of functions using the Lebesgue convergence theorem. This completes the proof of the theorem. □

The m-adic topology differs considerably from the usual topology on the interval $[0,1[$, but the Lebesgue measure (λ) is also translation invariant under the new operation. To show this statement we introduce the following notation. Let f be a complex function defined in the interval $[0,1[$ and denote by τ the *left translation operator* under the new operation, so

$$(\tau_y f)(x) := f(y \odot x) \qquad (x, y \in [0,1[), \tag{1.6}$$

and denote the *left translation of the set* E by

$$\tau_y(E) := \{y \odot x : x \in E\} \qquad (E \subseteq [0,1[,\ y \in [0,1[). \tag{1.7}$$

Theorem 1.5.2. *Let f be a complex function defined on the interval $[0,1[$, then*

(a) If the function f is Lebesgue integrable then $\tau_y f$ is also Lebesgue integrable and

$$\int_0^1 (\tau_y f)(x)\,dx = \int_0^1 f(x)\,dx \qquad (y \in [0,1[).$$

(b) In particular for all $E \subseteq [0,1[$ Lebesgue measurable set

$$\lambda(\tau_y(E)) = \lambda(E) \qquad (y \in [0,1[).$$

Proof. From (1.2), (1.3) and (1.4), using the translation invariant property of the measure μ we have

$$\int_0^1 (\tau_y f)(x)\,dx = \int_0^1 f(y \odot x)\,dx = \int_G f(y \odot |x|)\,d\mu =$$
$$= \int_G f(|\rho(y)x|)\,d\mu = \int_G f(|x|)\,d\mu = \int_0^1 f(x)\,dx.$$

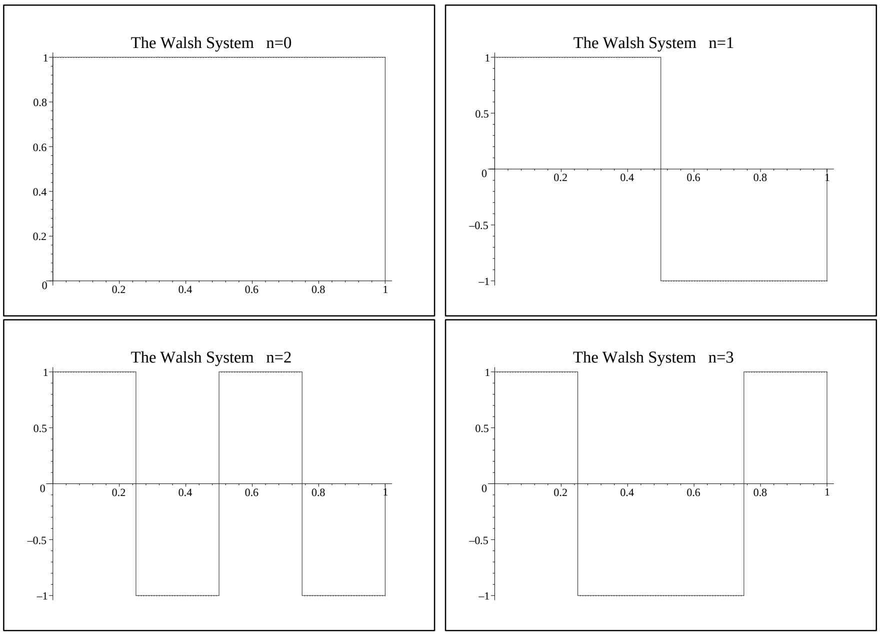
This completes the proof of the theorem. □

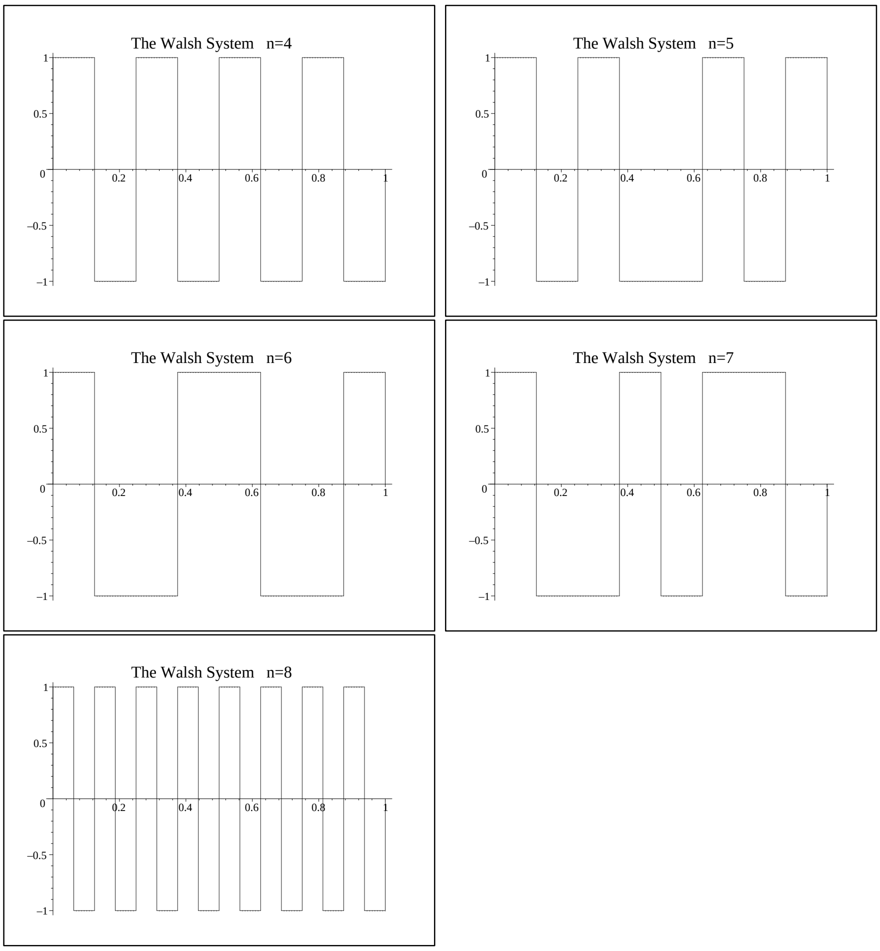
Finally, we represent the system ψ on the interval $[0,1]$ substituting it by the

$$\upsilon_n := \psi_n \circ \rho \qquad (n \in \mathbf{N})$$

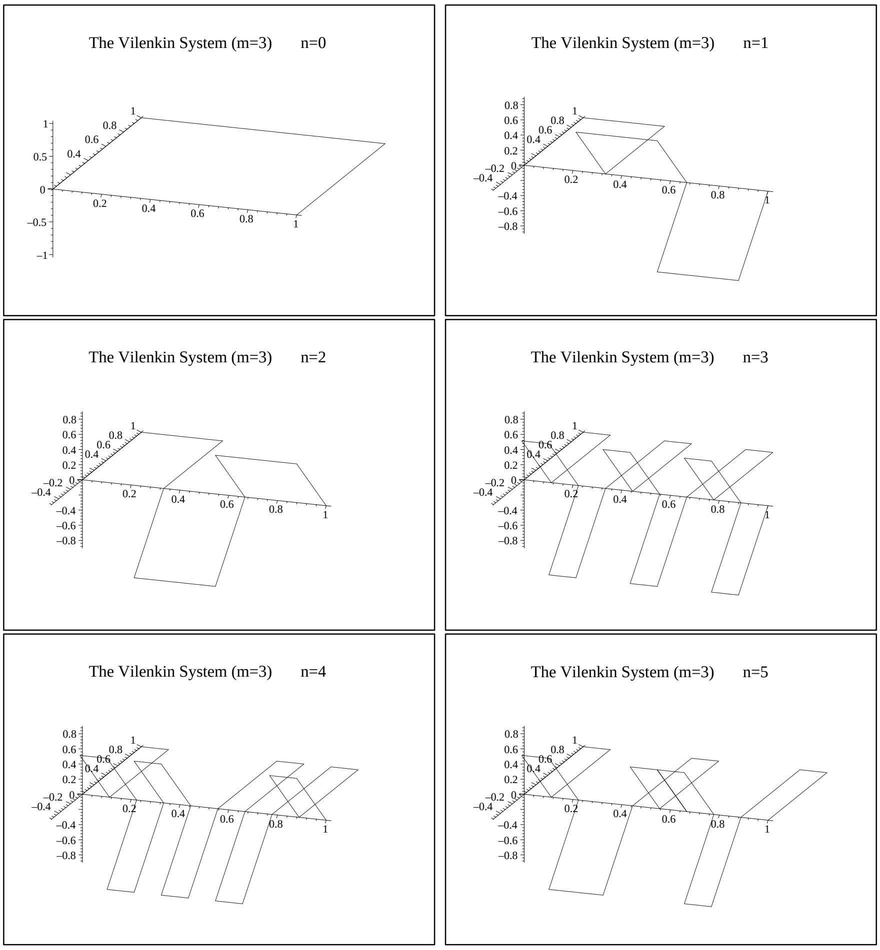
system, according to Theorem 1.5.1. In all cases we use the order of the system φ given in the examples of the Section 1.4.

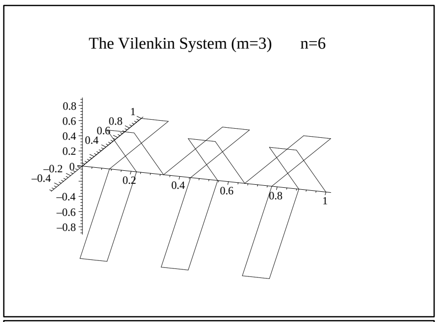
The Walsh system takes the values 1 and -1 only.

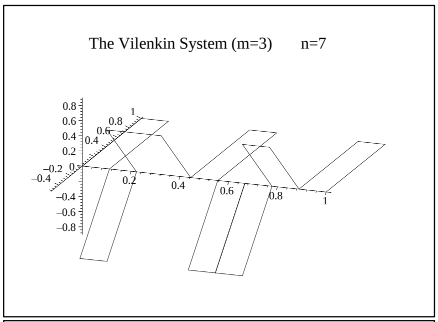


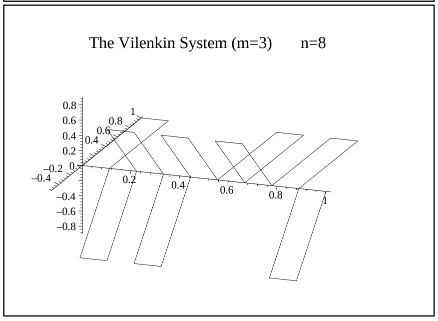

The Walsh System n=4
The Walsh System n=5
The Walsh System n=6
The Walsh System n=7
The Walsh System n=8
1
0.5
0
–0.5
–1
0.2
0.4
0.6
0.8
1

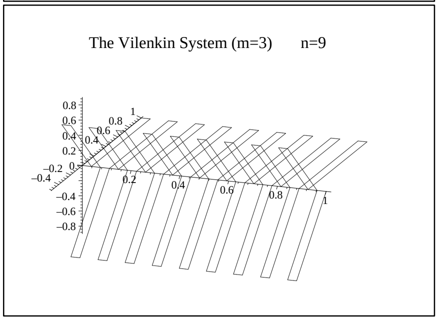
However, the Vilenkin system takes the values of the complex unit roots.



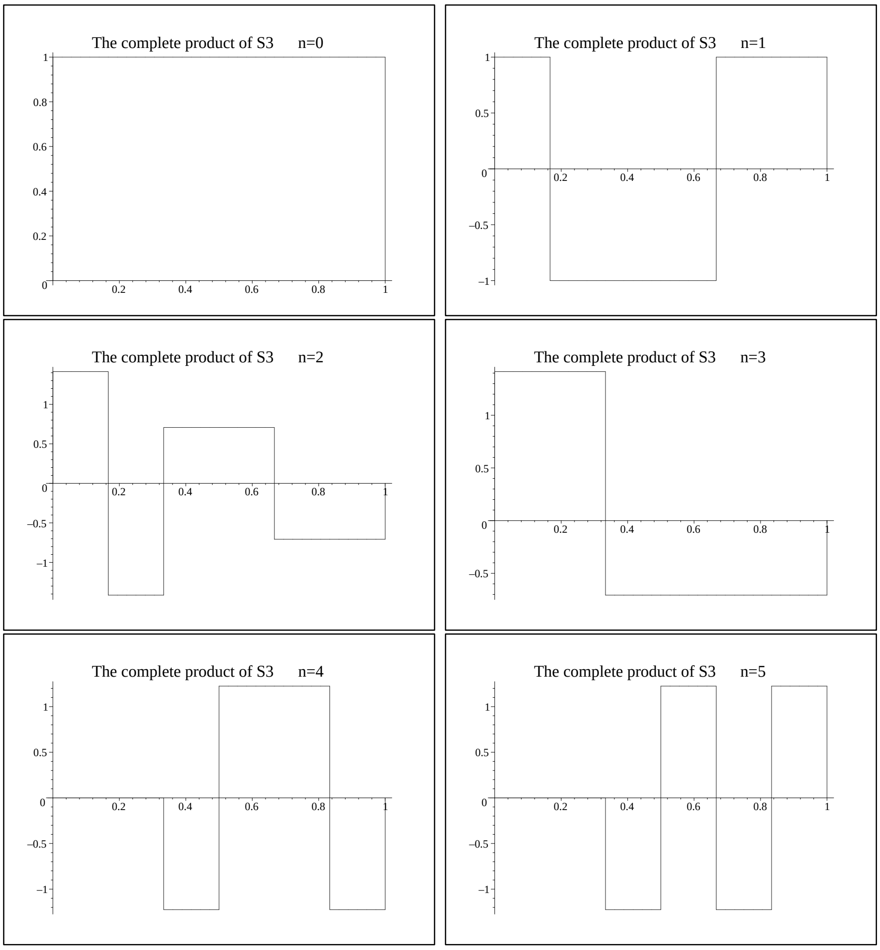

The Vilenkin System (m=3) n=6

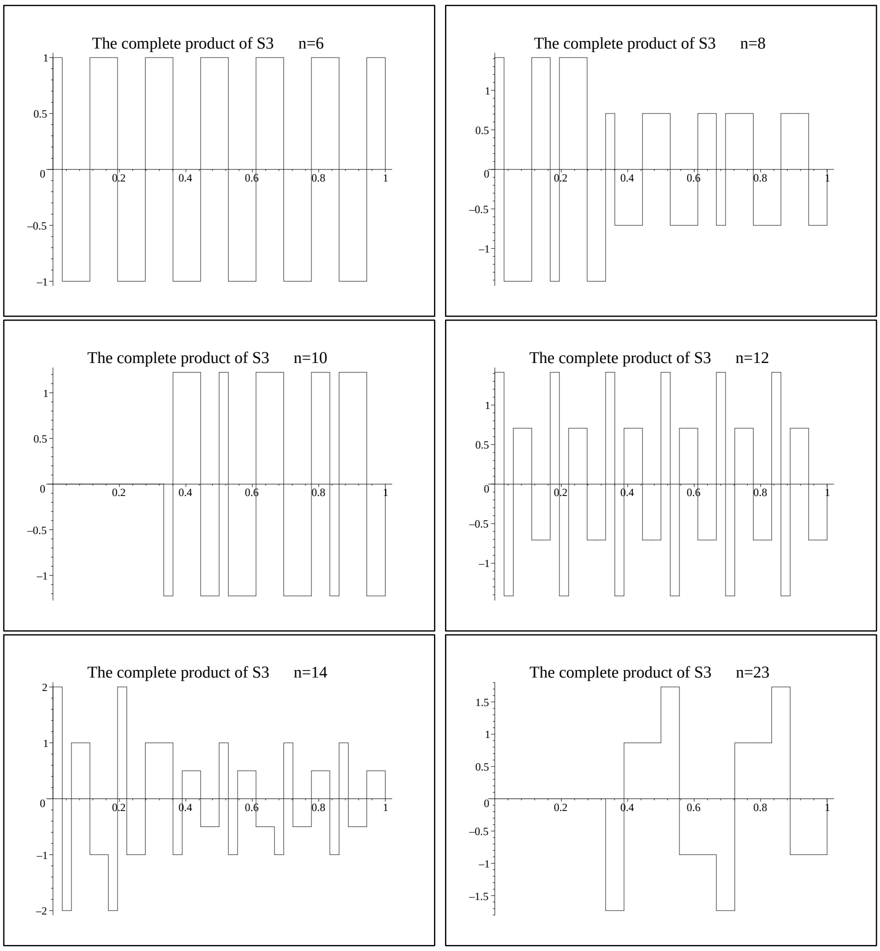

The Vilenkin System (m=3) n=7


The Vilenkin System (m=3) n=8

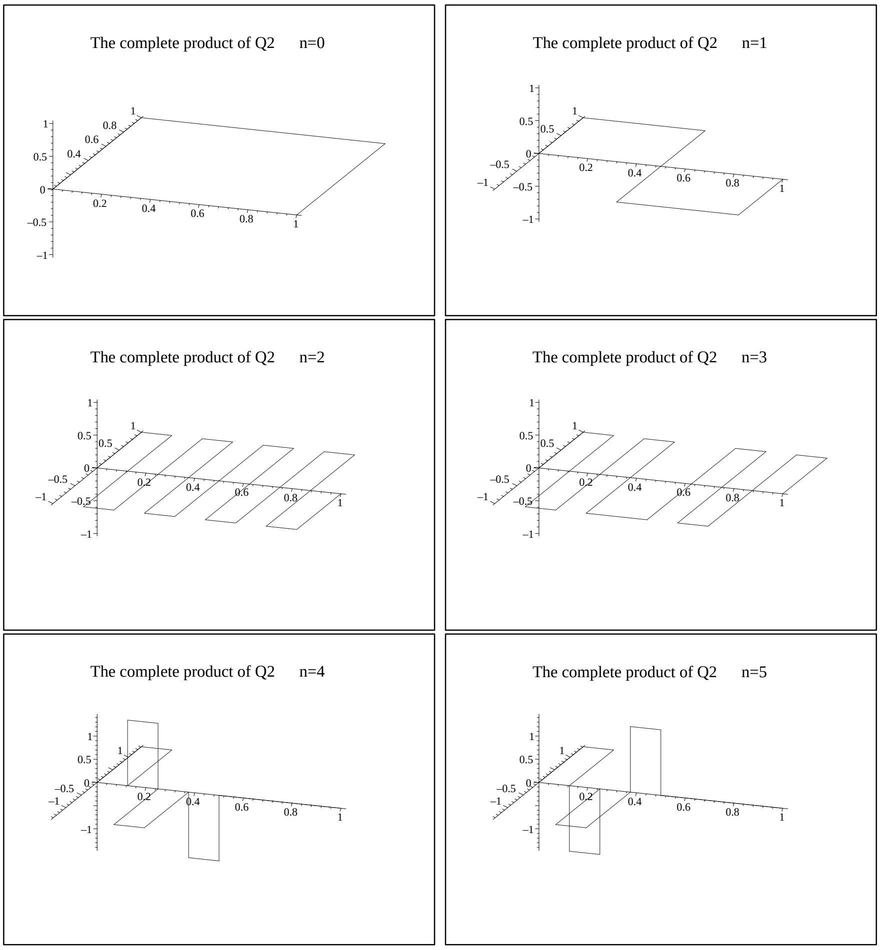

The Vilenkin System (m=3) n=9

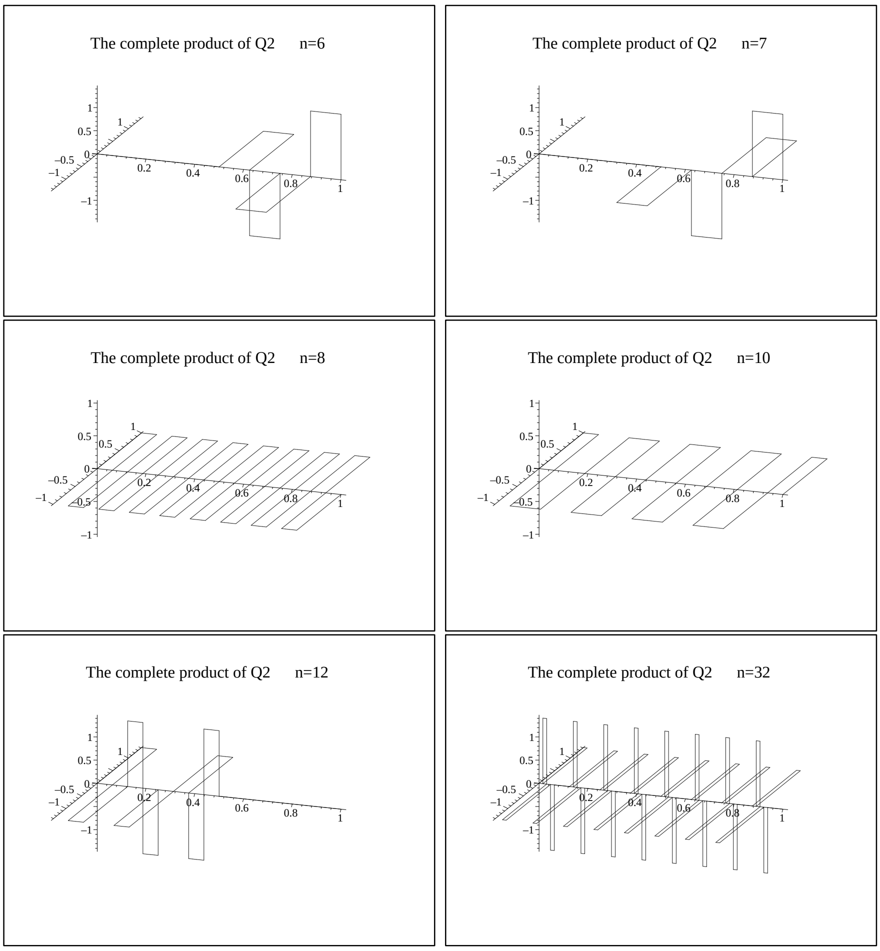
The system on the complete product of $\mathcal{S}_3$ takes only real values.




The complete product of S3 n=6
The complete product of S3 n=8
The complete product of S3 n=10
The complete product of S3 n=12
The complete product of S3 n=14
The complete product of S3 n=23

Finally, we can observe the system on the complete product of $\mathcal{Q}_2$ takes the values of the complex 4-th unit roots and zero.




The complete product of Q2 n=6
The complete product of Q2 n=7
The complete product of Q2 n=8
The complete product of Q2 n=10
The complete product of Q2 n=12
The complete product of Q2 n=32

Chapter 2

L^p-norm convergence of Fourier series and Fejér means

In Section 2.1 we introduce essential concepts in the study of Fourier analysis as the concept of L^p spaces, Fourier coefficients, Fourier series and Dirichlet kernels. The Dirichlet kernels play a prominent role in the convergence of Fourier series. The lemmas proved in this section will be used in this regard and they appeared first in [10] (see also [33]). Paley lemma is used to prove that the partial sum of Fourier series of every integrable function f defined on G have a subsequence converging to f in L^p-norm ($p \geq 1$) and a.e. This statement show us a notable difference with respect to the classic Fourier analysis.

Paley in [19] proved that the nth partial sum operators are bounded, uniformly in n, from $L^p(G)$ into itself for $1 < p < \infty$. It is equivalent to convergence of this operators in $L^p(G)$ norm for $1 < p < \infty$. This statement is known as the Paley's theorem. Paley's theorem was shown independently for arbitrary Vilenkin systems by Young [39], Schipp [21] and Simon [26]. We cannot generalize this statement for every non-abelian group. In Section 2.2 we show this negative result for a bounded group G with unbounded Ψ sequence. This result appears in [10] for the complete product of $\mathcal{S}_3$.

In Section 2.2 we also study the case $p = 1$. For an arbitrary group G there is a function $f \in L^1$, such that the sequence of partial sums $S_n f$ of Fourier series of f does not converge to the function f in L^1-norm. It is a well known result for Vilenkin groups (see [25], [26], [21] and [39]). However, a certain assumption for the modulus of continuity implies the L^1-norm convergence of Fourier series. This results appeared in [34] are the generalization of Simon's results in [27] for not necessarily commutative groups. The concept of modulus of continuity is due to Fine [3] and Morgenthaler [15] for the Walsh group.

Finally in Section 2.3 we prove the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case. The method of the proof is similar to [6]. For the Fejér kernels in the case of Abelian groups see also [1].

2.1 Fourier series and Dirichlet kernels

For $0 < p < \infty$ let $L^p(G)$ represent the set of all functions $L^0(G)$ such that

$$\|f\|_p := \left(\int_G |f|^p \, d\mu \right)^{\frac{1}{p}}$$

is finite. Similarly $L^\infty(G)$ represent the set of all functions $L^0(G)$ such that

$$\|f\|_\infty := \inf\{y \in \mathbf{R} : |f(x)| \leq y \text{ for a.e. } x \in G\}$$

is finite. Since the measure μ is finite the relation

$$L^\infty(G) \subset L^q(G) \subset L^p(G) \subset L^1(G) \qquad (1 < p < q < \infty)$$

is valid. For this reason considering the fact that $\|f\|_p$ is only a norm for $p \geq 1$, the most extensive set of functions on G we consider is just $L^1(G)$.

For $f \in L^1(G)$ we define the *Fourier coefficients* by

$$\widehat{f}_k := \int_G f\overline{\psi}_k d\mu \qquad (k \in \mathbf{N}),$$

and *the n-th partial sums of Fourier series* by

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{P}).$$

The *Dirichlet kernels* are defined as follows:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x,y) d\mu(y). \tag{2.1}$$

(2.1) mutates the importance of the Dirichlet kernels in the study of the convergence of Fourier series. The lemmas proved below are used in this regard. First we introduce the following notation. Every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the *expansion of n* with respect to the sequence m. In this case let $n^* = (n_0, n_1, \dots) \in G$. We often also use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and $n_{(k)} := \sum_{j=0}^{k-1} n_k M_k$, $n^{(k)} = \sum_{j=k}^{\infty} n_k M_k$.

Lemma 2.1.1. *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_n(x,y) = \sum_{k=0}^{\infty} D_{M_k}(x,y) \left(\sum_{s=0}^{n_k - 1} \varphi_k^s(x_k) \overline{\varphi}_k^s(y_k) \right) \psi_{n^{(k+1)}}(x) \overline{\psi}_{n^{(k+1)}}(y),$$

where $(n_0, n_1, ...)$ is the expansion of n and $x = (x_0, x_1, ...)$, $y = (y_0, y_1, ...)$.

Proof. For each $n \in \mathbf{N}$, $x, y \in G$, we have

$$D_n(x,y) = D_{M_{|n|}}(x,y)\left(\sum_{s=0}^{n_{|n|}-1} \varphi_{|n|}^s(x_{|n|})\overline{\varphi}_{|n|}^s(y_{|n|})\right) + \\ + \varphi_{|n|}^{n_{|n|}}(x_{|n|})\overline{\varphi}_{|n|}^{n_{|n|}}(y_{|n|})D_{n_{(|n|)}}(x,y).$$

By induction and applying that $\varphi_k^{n_k} \equiv 1$ for $k > |n|$ we prove the lemma. □

Lemma 2.1.2. *(Paley lemma) If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

Proof. For every positive integer n and $x, y \in G$ we have

$$D_{M_n}(x,y) = \prod_{k=0}^{n-1} \sum_{s=0}^{m_k-1} \varphi_k^s(x_k)\overline{\varphi}_k^s(y_k), \qquad D_{M_0} \equiv 1.$$

Then it is sufficient to prove that

$$\sum_{s=0}^{m_k-1} \varphi_k^s(x_k)\overline{\varphi}_k^s(y_k) = \begin{cases} m_k, & \text{for } x_k = y_k \\ 0, & \text{for } x_k \neq y_k \end{cases} \tag{2.2}$$

for each $k \in \mathbf{N}$. In other words, it is sufficient to demonstrate that for every finite and compact group G of order m:

$$\sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} d_\sigma u_{ij}^{(\sigma)}(x)\overline{u}_{ij}^{(\sigma)}(y) = \begin{cases} m, & \text{for } x = y \\ 0, & \text{for } x \neq y \end{cases} \qquad (x, y \in G).$$

Using the

$$\overline{u}_{i,j}^{(\sigma)}(x) = u_{ij}^{(\sigma)}(x^{-1}),$$
$$u_{i,j}^{(\sigma)}(xy) = \sum_{r=1}^{d_\sigma} u_{ir}^{(\sigma)}(x)u_{rj}^{(\sigma)}(y)$$

equalities for $x, y \in G$, $i, j \in \{1, ..., d_\sigma\}$, $\sigma \in \Sigma$, which are well known in representations theory (27.5 in [14]), we can state

$$\sum_{\sigma\in\Sigma}\sum_{i,j=1}^{d_\sigma} d_\sigma u_{ij}^{(\sigma)}(x)\overline{u}_{ij}^{(\sigma)}(y) = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} u_{ij}^{(\sigma)}(x)u_{ji}^{(\sigma)}(y^{-1})$$
$$= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma} d_\sigma u_{ii}^{(\sigma)}(xy^{-1}) = \sum_{\sigma\in\Sigma} d_\sigma \chi_\sigma(xy^{-1}),$$

where χ_σ is called the *character of the representation* $U^{(\sigma)}$. Since the above sum is the identity element of convolution (Theorem 27.41 in [14]), we have

$$\sum_{\sigma\in\Sigma} d_\sigma\chi_\sigma(xy^{-1}) = \begin{cases} m, & \text{for } x = y \\ 0, & \text{for } x \neq y \end{cases} \qquad (x \in G).$$

This completes the proof of the lemma. □

The Paley lemma is used to prove that the $S_{M_n}f$ partial sequence of Fourier sums converge to f in L^p-norm and a.e., if $f \in L^p(G)$, $p \geq 1$. Indeed, the

$$S_{M_n}f(x) = \int_G f(y)D_{M_n}(x,y)d\mu(y) = \frac{1}{\mu(I_n(x))}\int_{I_n(x)} f d\mu$$

operator is the conditional expectation with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in G$. Thus, the following statement is a consequence of the martingale convergence theorem (see [17]).

Corollary 2.1.3. *For each $f \in L^p(G)$, $p \geq 1$, $S_{M_n}f$ converges to f in L^p-norm, and a.e.*

The Weyl-Peter's theorem and the Theorem (27.43) in [14] secure that the system ψ is orthonormal and complete on $L^2(G)$. The definition of the system ψ secure all of function of ψ and their finite complex linear combinations that is all of representative functions are finite linear combinations of characteristics functions of intervals. Conversely, from the Paley lemma we have the characteristics function of an interval and the finite complex linear combinations of them all are representative functions. Since the set of all finite linear combinations of characteristics functions of intervals is dense in $L^1(G)$, we can state:

Corollary 2.1.4. *The system ψ is orthonormal, and complete in $L^1(G)$.*

2.2 L^p-norm convergence of Fourier series

According to the theorem of Banach-Steinhauss, $S_nf \to f$ as $n \to \infty$ in L^p norm for $f \in L^p(G)$ if and only if there exists a $C_p > 0$ such that $\|S_nf\|_p \leq C_p\|f\|_p$, $f \in L^p(G)$. Thus, we say that the S_n operators are of *type* (p,p). Since system ψ forms an orthonormal base in the Hilbert space $L^2(G)$, it is obvious that S_n is of type $(2,2)$.

For Vilenkin systems the operator S_n is of type (p,p) $(1 < p < \infty)$. In general this statement is not true. In the following theorem we suppose, if a finite group appear in the product of G it has the same system φ in the all of it's occurrences.

Theorem 2.2.1. *If G is a bounded group with unbounded sequence Ψ, then there is a $1 < p < 2$ for which the operator S_n is not of type (p,p).*

Proof. To prove this theorem we first observe that according to the Hölder's inequality for all normalized coordinate functions φ_k^s defined on the groups G_k we have

$$\|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \geq \|\varphi_k^s\|_2^2 = 1 \qquad (0 \leq s < m_k).$$

We distinguish two kind of groups G_k according as

$$\max_{s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty > 1 \qquad \text{or} \qquad \max_{s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty = 1.$$

If for a $k \in \mathbf{N}$, the group G_k satisfies the first condition, we choose $i_k < m_k$ for which the normalized coordinate function $\varphi_k^{i_k}$ of the group G_k satisfies

$$\|\varphi_k^{i_k}\|_1 \|\varphi_k^{i_k}\|_\infty = \max_{s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty.$$

In addition, denote by a_k an element of G_k for which $\varphi_k^{i_k}(a_k) = \|\varphi_k^{i_k}\|_\infty$, and $f_k \in L^1(G_k)$ by

$$f_k(x) = \begin{cases} 1, & \text{for } x = a_k \\ 0, & \text{for } x \neq a_k \end{cases} \qquad (x \in G_k).$$

The second case is more simple. Let $i_k = 0$ ($\varphi_k^{i_k} \equiv 1$) and also $f_k \equiv 1$ if for a $k \in \mathbf{N}$ the group G_k satisfies the second condition.

Thus for the two cases we have:

$$\left| \int_{G_k} f_k \overline{\varphi}_k^{i_k} \, d\mu_k \right| \|\varphi_k^{i_k}\|_1 = \|\varphi_k^{i_k}\|_1 \|\varphi_k^{i_k}\|_\infty \|f_k\|_1, \tag{2.3}$$

and

$$\left| \int_{G_k} f_k \overline{\varphi}_k^{i_k} \, d\mu_k \right| \|\varphi_k^{i_k}\|_2 \leq \|f_k\|_2. \tag{2.4}$$

Since the $\|f\|_p$ is a continuous function with respect to p for each $f \in L^p(G_k)$, the function

$$\Phi_k(p) := \left| \int_{G_k} f_k \overline{\varphi}_k^{i_k} \, d\mu_k \right| \frac{\|\varphi_k^{i_k}\|_p}{\|f_k\|_p} \qquad (1 \le p \le 2)$$

is continuous with respect to p, and for the first case by (2.3) and (2.4) we obtain $\Phi_k(1) = \|\varphi_k^{i_k}\|_1 \|\varphi_k^{i_k}\|_\infty > 1$ and $\Phi_k(2) < 1$. The continuity of Φ_k assure that there is a $1 < p_k < 2$ so that

$$\left| \int_{G_k} f_k \overline{\varphi}_k^{i_k} \, d\mu_k \right| \|\varphi_k^{i_k}\|_p \ge \sqrt{\|\varphi_k^{i_k}\|_1 \|\varphi_k^{i_k}\|_\infty} \|f_k\|_p \tag{2.5}$$

for all $1 \le p \le p_k$. In the second case the value of p_k is 2.

The boundedness of the group G implies there are only finite different values of p_k $(k \in \mathbf{N})$, hence for $p := \min_{k \in \mathbf{N}} \{p_k\} > 1$ the statement (2.5) is valid.

If j is an arbitrary positive integer and $n = \sum_{k=0}^{j-1} i_k M_k$, then define $F_j \in L^p(G)$ by

$$F_j(x) := \prod_{k=0}^{j-1} f_k(x_k) \qquad (x = (x_0, x_1, \dots) \in G).$$

Since $\|F_j\|_p = \prod_{k=0}^{j-1} \|f_k\|_p$ it follows that

$$\begin{aligned} \|S_{n+1}F_j - S_n F_j\|_p &= \left| \int_G F_j \overline{\psi}_n d\mu \right| \|\psi_n\|_p = \\ &= \prod_{k=0}^{j-1} \left| \int_{S_3} f_k \overline{\varphi}_k^s d\mu_k \right| \|\varphi_k^s\|_p \ge \sqrt{\Psi_k} \|F_j\|_p, \end{aligned} \tag{2.6}$$

but if S_n is of type (p,p) then there is $C_p > 0$ so that

$$\|S_{n+1}F_j - S_nF_j\|_p \leq \|S_{n+1}F_j\|_p + \|S_nF_j\|_p \leq 2C_p\|F_j\|_p$$

for each $j > 0$, which contradict (2.6) because the sequence Ψ is not bounded. For this reason operators S_n are not uniformly of type (p,p). This completes the proof of the theorem. □

We restrict our attention to the case $p = 1$.

Theorem 2.2.2. *For an arbitrary group G there exists a function $f \in L^1(G)$ such that the sequence of partial sums S_nf of the Fourier series of f does not converge to the function f in L^1-norm.*

Proof. It will be sufficient to show that the operators S_n are not uniformly bounded in L^1-norm for all $n \in \mathbf{N}$. Let $f_n(x) = D_{M_n}(x,e)$, where e is the identity element of G. Thus $\|f_n\|_1 = 1$ for all $n \in \mathbf{N}$. We distinguish two cases according as the sequence Ψ is bounded or not.

If the sequence Ψ is bounded there exists a $C > 0$ such that

$$\|\varphi_r^s\|_1\|\varphi_r^s\|_\infty < C \qquad (r \in \mathbf{N},\ 0 \leq s < m_r). \tag{2.7}$$

We construct the sequence ς_j $(j \in \mathbf{N})$ as follows: set $\varsigma_0 = 0$ and let ς_j be the least number greater than ς_{j-1} such that

$$C\sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} d_r^* < \frac{1}{4}M_{\varsigma_j} \qquad (j \in \mathbf{N}), \tag{2.8}$$

where d_r^* is the greatest dimension of a representation of the group G_r $(r \in \mathbf{N})$. We can found always this kind of number j because d_r^* is a divisor of m_k, thus the quotient

$$\prod_{r=\varsigma_l+1}^{s} \frac{m_r}{d_r^*} \to \infty, \quad \text{if } l \text{ is fix and } s \to \infty.$$

The sequence ς is used to define the number k. Take the coordinates of the expansion of k in the following manner. Let k_{ς_j} $(j \in \mathbf{N},\ \varsigma_j < n)$ be the least natural number greater than 0 such that $\varphi_{\varsigma_j}^{k_{\varsigma_j}}(e) \neq 0$ and let the others coordinates be 0. Then

$$\sum_{s=0}^{k_l-1} \varphi_l^s(x_l)\overline{\varphi}_l^s(e) = \begin{cases} 0 & \text{if } k_l = 0 \\ 1 & \text{if } k_l \neq 0 \end{cases} \qquad (x_l \in G_l). \tag{2.9}$$

Note that the coordinates of the expansion of k is not 0 if the index of the coordinates is in the range of the sequence ς and less than n.

Since $k < M_n$ by Lemmas 2.1.1 and 2.1.2 we obtain

$$\begin{aligned}
&\|S_k f_n\|_1 = \int_G |D_k(x,e)| dx \\
&= \sum_{j=0}^{\infty} \int_{I_j \setminus I_{j+1}} \left| \sum_{l=0}^{\infty} D_{M_l}(x,e) \left(\sum_{s=0}^{k_l-1} \varphi_l^s(x_l)\overline{\varphi}_l^s(e) \right) \psi_{k^{(l+1)}}(x)\overline{\psi}_{k^{(l+1)}}(e) \right| dx \\
&= \sum_{j=0}^{n-1} \int_{I_j \setminus I_{j+1}} \left| \sum_{l=0}^{j} M_l \left(\sum_{s=0}^{k_l-1} \varphi_l^s(x_l)\overline{\varphi}_l^s(e) \right) \psi_{k^{(l+1)}}(x)\overline{\psi}_{k^{(l+1)}}(e) \right| dx.
\end{aligned}$$

In addition, we denote by p the greatest natural number for which $\varsigma_p < n$ is true. Thus by 2.9 and by the definition of the number k we have

$$\begin{aligned}
\|S_k f_n\|_1 &= \sum_{j=0}^{p} \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \left| \sum_{l=0}^{j} M_{\varsigma_l} \psi_{k^{(\varsigma_l+1)}}(x)\overline{\psi}_{k^{(\varsigma_l+1)}}(e) \right| dx \\
&= \sum_{j=0}^{p} \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \left| M_{\varsigma_j} + \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} |\varphi_r^{k_r}(e)|^2 \varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j})\overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e) \right| \\
&\qquad\qquad \times |\psi_{k^{(\varsigma_j+1)}}(x)||\overline{\psi}_{k^{(\varsigma_j+1)}}(e)|\, dx =
\end{aligned}$$

$$= \sum_{j=0}^{p} \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \left| M_{\varsigma_j} + \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} |\varphi_r^{k_r}(e)|^2 \varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j}) \overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e) \right| dx$$
$$\times \int_G |\psi_{k^{(\varsigma_j+1)}}(x)| |\overline{\psi}_{k^{(\varsigma_j+1)}}(e)| \, dx.$$

The condition for k implies that $|\varphi_r^{k_r}(e)| = \|\varphi_r^{k_r}\|_\infty$ ($r \in \mathbf{N}$), thus the last integral can be written as

$$\int_G |\psi_{k^{(\varsigma_j+1)}}(x)| |\overline{\psi}_{k^{(\varsigma_j+1)}}(e)| \, dx = \|\psi_{k^{(\varsigma_j+1)}}\|_\infty \|\psi_{k^{(\varsigma_j+1)}}\|_1 \geq 1.$$

On the other hand,

$$\sum_{j=0}^{p} \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \left| M_{\varsigma_j} + \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} |\varphi_r^{k_r}(e)|^2 \varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j}) \overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e) \right| dx$$
$$\geq \sum_{j=0}^{p} \left(\int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} M_{\varsigma_j} \, dx \right.$$
$$\left. - \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \left| \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} |\varphi_r^{k_r}(e)|^2 \varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j}) \overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e) \right| dx \right)$$

Combining this estimate with the earlier one, by 2.7 and 2.8 we conclude that

$$\|S_k f_n\|_1 \geq \sum_{j=0}^{p} \left(M_{\varsigma_j} \left(\frac{1}{M_{\varsigma_j}} - \frac{1}{M_{\varsigma_j+1}} \right) \right.$$
$$\left. - \int_{I_{\varsigma_j} \setminus I_{\varsigma_j+1}} \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} d_r^* |\varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j})| |\overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e)| \, dx \right)$$

$$\geq \sum_{j=0}^{p} \left(\frac{m_{\varsigma_j}-1}{m_{\varsigma_j}} - \frac{1}{M_{\varsigma_j}} \int_{G_{\varsigma_j}\setminus\{e\}} \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} d_r^* |\varphi_{\varsigma_j}^{k_{\varsigma_j}}(x_{\varsigma_j})||\overline{\varphi}_{\varsigma_j}^{k_{\varsigma_j}}(e)|\, dx_{\varsigma_j} \right)$$

$$\geq \sum_{j=0}^{p} \left(\frac{1}{2} - \frac{1}{M_{\varsigma_j}} \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} d_r^* \|\varphi_{\varsigma_j}^{k_{\varsigma_j}}\|_1 \|\varphi_{\varsigma_j}^{k_{\varsigma_j}}\|_\infty \right)$$

$$\geq \sum_{j=0}^{p} \left(\frac{1}{2} - \frac{1}{M_{\varsigma_j}} C \sum_{l=0}^{j-1} M_{\varsigma_l} \prod_{r=\varsigma_l+1}^{\varsigma_j-1} d_r^* \right) \geq \sum_{j=0}^{p} \left(\frac{1}{2} - \frac{1}{M_{\varsigma_j}} \frac{1}{4} M_{\varsigma_j} \right)$$

$$= \sum_{j=0}^{p} \frac{1}{4} \to \infty \qquad \text{if } n \to \infty.$$

If the sequence Ψ is not bounded the definition of k is more simple. Take the coordinates of the expansion of k in the following manner. Let k_j $(j \in \mathbf{N},\ j < n)$ be the least natural number greater than 0 such that $\varphi_j^{k_j}(e) \neq 0$ and let k_j $(j \in \mathbf{N},\ j \geq n)$ be 0. Then

$$\sum_{s=0}^{k_l-1} \varphi_l^s(x_l)\overline{\varphi}_l^s(e) = \begin{cases} 1 & \text{if } l < n \\ 0 & \text{if } l \geq n \end{cases} \qquad (x_l \in G_l). \tag{2.10}$$

Since $k < M_n$ by Lemmas 2.1.1 and 2.1.2 we obtain

$$\|S_k f_n\|_1 = \int_G |D_k(x,e)|\, dx$$

$$= \sum_{j=0}^{n-1} \int_{I_j\setminus I_{j+1}} \left| \sum_{l=0}^{j} M_l \left(\sum_{s=0}^{k_l-1} \varphi_l^s(x_l)\overline{\varphi}_l^s(e) \right) \psi_{k^{(l+1)}}(x)\overline{\psi}_{k^{(l+1)}}(e) \right| dx$$

$$\geq \int_{I_0\setminus I_1} M_0 \left|\psi_{k^{(1)}}(x)\overline{\psi}_{k^{(1)}}(e)\right|\, dx = \frac{m_0-1}{m_0} \|\psi_{k^{(1)}}\|_1 \|\psi_{k^{(1)}}\|_\infty$$

$$\geq \frac{1}{2} \frac{\Psi_n}{|\varphi_0\|_1 \|\varphi_0\|_\infty} \to \infty \qquad \text{if } n \to \infty.$$

This completes the proof of the theorem. □

Let $f \in L^p(G)$, $1 \le p \le \infty$ and I an interval. Then $I = I_n(x)$ for some $x \in G, n \in \mathbf{N}$. Denote by
(2.11)

$$\omega^{(p)}(f, I) := \sup_{h \in I_n} \left(\frac{1}{\mu(I)} \int_I |\tau_h f - f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad \omega(f, I) := \omega^{(1)}(f, I)$$

the *local modulus of continuity of* f on I and

$$\text{(2.12)} \quad \omega_n^{(p)}(f) := \sup_{h \in I_n} \|\tau_h f - f\|_p, \quad (n \in \mathbf{N}), \qquad \omega_n(f) := \omega_n^{(1)}(f)$$

the *n-th modulus of continuity of* f on L^p, where $\tau_h f(x) := f(xh)$ is the right translation operator. We remark that if we use the left translation operator, we obtain identical value for the modulus of continuity, because the measure is both left and right translation invariant and I_n is a normal subgroup of G. Notice that $\omega_n^{(p)}(f) \searrow 0$, $n \to \infty$ and $\omega_n^{(p)}(f)$ increases when p is also increases.

We often use the following lemma in the proof of the theorems.

Lemma 2.2.3. *Let Σ be the dual object of a finite compact group G and $\{\varphi_k : 0 \le k < |G|\}$ be the set of all normalized coordinate functions of the group G. Then for every $x, y \in G$ and $0 \le j < |G|$ we have*

$$\left| \sum_{k=0}^{j} \varphi_k(x) \overline{\varphi_k}(y) \right| \le |G|.$$

Proof. For $\sigma \in \Sigma$ let A_σ be the collection of all numbers k in $\{0, 1, \dots j\}$ such that φ_k is a normalized coordinate function corresponding to the chosen representation of σ. Using the inequality of Cauchy-Buniakovskii and the unitary property of the matrices appeared in the representations, we have

$$\begin{aligned}
\left|\sum_{k=0}^{j}\varphi_k(x)\overline{\varphi_k}(y)\right| &\le \sum_{\sigma\in\Sigma}\left|\sum_{k\in A_\sigma}\varphi_k(x)\overline{\varphi_k}(y)\right| \\
&\le \sum_{\sigma\in\Sigma}\left(\sum_{k\in A_\sigma}|\varphi_k(x)|^2\right)^{\frac{1}{2}}\left(\sum_{k\in A_\sigma}|\varphi_k(y)|^2\right)^{\frac{1}{2}} \\
&\le \sum_{\sigma\in\Sigma}\left(d_\sigma^2\right)^{\frac{1}{2}}\left(d_\sigma^2\right)^{\frac{1}{2}} \\
&= \sum_{\sigma\in\Sigma} d_\sigma^2 = |G|.
\end{aligned}$$

This completes the proof of the lemma. □

We prove that a certain assumption for the modulus of continuity implies the L^1 convergence of Fourier series. The following theorem was proved by Simon in [27] and we follow the method of his paper.

Theorem 2.2.4. *Let f be a function in $L^1(G)$ for which the following condition holds:*

$$\omega_k(f) = o(\Psi_k \sum_{j=0}^{k} m_j)^{-1}. \tag{2.13}$$

Then the sequence of partial sums $S_n f$ of Fourier series of f converges in L^1-norm to f.

Proof. Let us write

$$n = jM_s + \sum_{l=0}^{s-1} n_l M_l \qquad (1 \le j < m_k)$$

for $n \in \mathbf{N}$. Then

$$\begin{aligned}
S_n f(x) &= \sum_{k=0}^{jM_s-1} \hat{f}_k \psi_k(x) + \sum_{k=jM_s}^{n-1} \hat{f}_k \psi_k(x) \\
&= S_{jM_s} f(x) + \int_G f(y) \sum_{k=jM_s}^{n-1} \overline{\psi}_k(y)\psi_k(x)dy \\
&= S_{jM_s} f(x) + \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s) \sum_{k=0}^{n^*-1} \overline{\psi}_k(y)\psi_k(x)dy \\
&= S_{jM_s} f(x) + \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s) D_{n^*}(x,y)dy,
\end{aligned}$$

where $n^* = n - jM_s$. From this follows that

(2.14)

$$\|S_n f - f\|_1 \le \|S_{jM_s} f - f\|_1 + \int_G \left| \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s) D_{n^*}(x,y)dy \right| dx.$$

By the fact

$$\int_G D_n(x,y)dy = \sum_{k=0}^{n-1} \psi_k(x) \int_G \overline{\psi_k}(y)dy = \psi_0(x) \equiv 1, \tag{2.15}$$

and Lemma 2.1.1 we have

$$\begin{aligned}
\|S_{jM_s} f - f\|_1 &= \int_G \left| \int_G f(y) D_{jM_s}(x,y)dy - \int_G f(x) D_{jM_s}(x,y)dy \right| dx \\
&\le \int_G \int_G |f(y) - f(x)||D_{jM_s}(x,y)|dy\, dx \\
&\le \int_G \int_G |f(y) - f(x)||D_{M_s}(x,y)| \left| \sum_{r=0}^{j-1} \varphi_s^j(x_s)\overline{\varphi_s^j}(y_s) \right| dy\, dx.
\end{aligned}$$

By Paley lemma and Lemma 2.2.3 we have

$$\begin{aligned}\|S_{jM_s}f - f\|_1 &\le m_s M_s \int_G \int_{I_s(x)} |f(y) - f(x)|dy\,dx \\ &= m_s M_s \int_G \int_{I_s} |f(xt) - f(x)|dt\,dx.\end{aligned}$$

Thus applying the Fubini's theorem we obtain

$$\|S_{jM_s}f - f\|_1 \le m_s M_s \int_{I_s} \int_G |f(xt) - f(x)|dt\,dx \le m_s \omega_s(f). \tag{2.16}$$

In order to estimate the second term in (2.14) let $h(p)$ be the element of G with expansion $h(p) = (\underset{\breve{0}}{e}, \underset{\breve{1}}{e}, \dots, \underset{s\breve{-}1}{e}, \underset{\breve{s}}{p}, \underset{s\breve{+}1}{e}, \dots)$. Then $h(p) \in I_s$. If $\varphi_s^{n_s}$ is a normalized coordinate function of the group G_s, then there exists a $\sigma \in \Sigma_s$, and $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_s^{n_s} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}.$$

Using the fact that the measure μ is both right and left translation invariant, $n^* < M_s$, and $D_{n^*}(x, y)$ does not depend on the s-th coordinate of y, we have

$$\begin{aligned}\int_G f(yh(p))\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy &= \int_G f(y)\overline{\varphi_s^j}(y_s p^{-1})D_{n^*}(x,y)dy \\ &= \int_G f(y)\sqrt{d_\sigma}\overline{u_{i,j}^{(\sigma)}}(y_s p^{-1})D_{n^*}(x,y)dy \\ &= \int_G f(y)\sqrt{d_\sigma}\sum_{r=1}^{d_\sigma} \overline{u_{i,r}^{(\sigma)}}(y_s)u_{j,r}^{(\sigma)}(p)D_{n^*}(x,y)dy \\ &= \sum_{r=1}^{d_\sigma} u_{j,r}^{(\sigma)}(p)\int_G f(y)\sqrt{d_\sigma}\overline{u_{i,r}^{(\sigma)}}(y_s)D_{n^*}(x,y)dy.\end{aligned}$$

The coordinate functions form an orthonormal system:

$$\sum_{p\in G_s} u_{j,r}^{(\sigma)}(p) = m_s \int_{G_s} u_{j,r}^{(\sigma)}(x)\, d\mu_s(x) = 0.$$

For this reason

$$\sum_{p\in G_s} \int_G f(yh(p))\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy = 0.$$

Denote by $\langle .,. \rangle$ the inner product (for the complex numbers $a+b\imath, c+d\imath$ $\langle a+b\imath, c+d\imath\rangle = ac+bd$). Thus

$$\int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy + \sum_{\substack{p\in G_s\\ p\neq e}} \int_G f(yh(p))\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy = 0,$$

$$\left|\int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy\right|^2 + \sum_{\substack{p\in G_s\\ p\neq e}} \left\langle \int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy, \right.$$
$$\left. \int_G f(yh(p))\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy \right\rangle = 0.$$

Then there exists a $p \in G_s$, $p \neq e$ such that

$$\left|\int_G f(yh(p))\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy - \int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy\right|$$
$$\geq \left|\int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy\right|.$$

Thus

$$\left|\int_G f(y)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy\right|$$
$$\leq \int_G |f(yh(p)) - f(y)||\varphi_s^j(y_s)||D_{n^*}(x,y)|dy,$$

and for the second term in (2.14) we obtain

$$\int_G \left| \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy \right| dx$$
$$\leq \int_G |\varphi_s^j(x_s)| \int_G |f(yh(p)) - f(y)||\varphi_s^j(y_s)||D_{n^*}(x,y)|dy\, dx.$$

Applying Fubini's theorem and by Lemmas 2.1.1, 2.1.2 and 2.2.3 we have

$$\int_G \left| \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy \right| dx$$
$$\leq \int_G |f(yh(p)) - f(y)||\varphi_s^j(y_s)| \int_G |\varphi_s^j(x_s)||D_{n^*}(x,y)|dx\, dy$$
$$\leq \sum_{k=0}^{s-1} \int_G |f(yh(p)) - f(y)||\overline{\psi_{n^{(k+1)}}}(y)|$$
$$\times \int_G D_{M_k}(x,y) \left| \sum_{s=0}^{n_k-1} \varphi_k^s(x_k)\overline{\varphi_k^s}(y_k) \right| |\psi_{n^{(k+1)}}(x)|dx\, dy$$
$$\leq \sum_{k=0}^{s-1} \int_G |f(yh(p)) - f(y)||\overline{\psi_{n^{(k+1)}}}(y)|m_k\|\psi_{n^{(k+1)}}\|_1 dy.$$

Consequently,

$$\int_G \left| \int_G f(y)\varphi_s^j(x_s)\overline{\varphi_s^j}(y_s)D_{n^*}(x,y)dy \right| dx \leq \Psi_s\omega_s(f)\sum_{k=0}^{s-1} m_k. \tag{2.17}$$

By (2.14), (2.16) and (2.17) we obtain

$$\|S_n f - f\|_1 \leq \Psi_s\omega_s(f)\sum_{k=0}^{s} m_k.$$

Theorem 2.2.4 is proved. □

In the case when the sequence m is bounded it is easy to see that there exists a $c > 0$ such that $m_k \le c \log m_k$ for all $k \in \mathbf{N}$. Then we obtain from Theorem 2.2.4:

Corollary 2.2.5. *Let f be a function in $L^1(G)$ for which the following condition holds:*

$$\omega_k(f) = o(\Psi_k \log M_k)^{-1}. \tag{2.18}$$

If the group G is a bounded group then the sequence of partial sums $S_n f$ of Fourier series of f converges in L^1-norm to f.

The above corollary is similar to the known Dini-Lipschitz test for uniform convergence of Vilenkin series [18]. In a similar manner to the proof of the Theorem 2.2.4 we have carried out similar calculations and got the following result:

Theorem 2.2.6. *Let f be a continuous function on G for which the following condition holds:*

$$\omega_k^\infty(f) = o(\Psi_k \sum_{j=0}^{k} m_k)^{-1}. \tag{2.19}$$

Then the sequence of partial sums $S_n f$ of Fourier series of f converges in uniform norm to f.

The above results are simpler if the sequence Ψ is bounded. In this case Ψ_k vanishes in (2.13), (2.18) and (2.19). Moreover, we have

Theorem 2.2.7. *Let f be a continuous function on G for which the following condition holds:*

$$\sum_{k=0}^{\infty} m_k \omega_k^\infty(f) < \infty, \tag{2.20}$$

and suppose the sequence Ψ is bounded. Then the sequence of partial sums $S_n f$ of Fourier series of f converges in uniform norm to f.

Proof. From Lemma 2.1.1 there follows for all $k_0 \in \mathbf{N}$ that

$$\begin{aligned}|S_n f(x) - f(x)| &= \left| \int_G (f(y) - f(x)) D_n(x,y) dy \right| \\ &\leq \sum_{k=0}^{k_0} \left| \int_G (f(y) - f(x)) D_{M_k}(x,y) \right. \\ &\quad \times \left. \left(\sum_{s=0}^{n_k - 1} \varphi_k^s(x_k) \overline{\varphi}_k^s(y_k) \right) \psi_{n^{(k+1)}}(x) \overline{\psi}_{n^{(k+1)}}(y) dy \right| \\ &\quad + \sum_{k=k_0+1}^{\infty} \int_G |f(y) - f(x)| D_{M_k}(x,y) \\ &\quad \times \left| \sum_{s=0}^{n_k - 1} \varphi_k^s(x_k) \overline{\varphi}_k^s(y_k) \right| |\psi_{n^{(k+1)}}(x)| |\overline{\psi}_{n^{(k+1)}}(y)| dy.\end{aligned}$$

If k_0 is fixed the first term is the sum of $k_0 + 1$ Fourier coefficients, so it converge to 0 when $n \to \infty$ (f is a continuous function). We can estimate the second sum from above for all $\varepsilon > 0$ by

$$\begin{aligned}&\sum_{k=k_0+1}^{\infty} m_k |\psi_{n^{(k+1)}}(x)| \int_G |f(y) - f(x)| D_{M_k}(x,y) || \overline{\psi}_{n^{(k+1)}}(y)| dy \leq \\ &\leq \sum_{k=k_0+1}^{\infty} m_k |\psi_{n^{(k+1)}}(x)| \|\psi_{n^{(k+1)}}\|_1 \omega_k^{\infty}(f) \leq c \sum_{k=k_0+1}^{\infty} m_k \omega_k^{\infty}(f) < \varepsilon\end{aligned}$$

if k_0 is a sufficiently great number. This completes the proof of the theorem. □

2.3 L^p-norm convergence of Fejér means

Denote *the Fejér means of Fourier series* by

$$\sigma_n f = \frac{1}{n}\sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbf{P}),$$

and the *Fejér kernels* by

$$K_n := \frac{1}{n}\sum_{k=1}^{n-1} D_k \qquad (n \in \mathbf{P}).$$

Then we have

$$\sigma_n f(x) = \int_G f(y) K_n(x,y) d\mu(y) \qquad (x \in G, n \in \mathbf{P}).$$

Lemma 2.3.1. *If G is a bounded group, then there is a $C > 0$ such that*

$$\sup_{x\in G}\int_G |K_n(x,y)|\, d\mu(y) \le C.$$

Proof. Throughout this proof $C > 0$ will denote an absolute constant which will not necessarily be the same at different occurrences. Let r be a fixed natural number. To estimate the kernels $|K_n|$ we prove that for every $r \in \mathbf{P}$

$$\sum_{j=0}^{r} M_j d_{n^{(j)}} \le \frac{\sqrt{2}}{\sqrt{2}-1} M_r d_{n^{(r)}}, \tag{2.21}$$

where $d_n = \prod_{k=0}^{\infty} d_k^{(n_k)}$ and $d_k^{(n_k)}$ is the dimension of the representation corresponding to $\varphi_k^{n_k}$. Set

$$b_j := M_j d_{n^{(j)}} \qquad (0 \le s \le r),$$

thus

$$b_{j+1} := M_{j+1} d_{n^{(j+1)}} = M_j d_{n^{(j)}} \frac{m_j}{d_j^{(n_j)}} \geq \sqrt{2} b_j$$

for $0 \leq s < r$, since $\left(d_j^{(n_j)}\right)^2 < m_j$. Then

$$\sum_{j=0}^{r} b_j \geq b_0 + \sqrt{2} \sum_{j=0}^{r} b_j - \sqrt{2} b_r.$$

Consequently

$$\sum_{j=0}^{r} b_j \leq \frac{\sqrt{2}}{\sqrt{2}-1} b_r.$$

This proves the inequality (2.21).

First we will estimate the absolute value of

$$K_{n^{(s)},M_s} := \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} D_a \qquad (s \in \mathbf{N})$$

kernels if $x \in G, y \in I_r(x) \setminus I_{r+1}(x)$ and applay the

$$nK_n = \sum_{s=0}^{|n|} \sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s,M_s} \qquad (n \in \mathbf{P}) \tag{2.22}$$

identity. Let $s \leq r$. Then by Lemma 2.1.1

$$K_{n^{(s)},M_s}(x,y) :=$$
$$= \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} \sum_{k=0}^{r} M_k \left(\sum_{j=0}^{a_k-1} \varphi_k^j(x_k) \overline{\varphi}_k^j(y_k) \right) \psi_{a^{(k+1)}}(x) \overline{\psi}_{a^{(k+1)}}(y),$$

where $x \in G$, $y \in I_r(x) \setminus I_{r+1}(x)$. Since G is a bounded group, by (2.21) we have

$$|K_{n^{(s)},M_s}(x,y)| \leq CM_sM_rd_{n^{(s)}}.$$

Then

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_{n^{(s)},M_s}(x,y)|d\mu(y) \leq CM_sd_{n^{(s)}}.$$

Next we turn our attention to the $s > r$ case. In this case it is necessary to set a better estimate of $|K_{n^{(s)},M_s}(x,y)|$.

$$\begin{aligned}
&K_{n^{(s)},M_s}(x,y) \\
&\qquad = \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} \sum_{k=0}^{r-1} M_k \left(\sum_{j=0}^{a_k-1} |\varphi_k^j(x_k)|^2 \right) \psi_{a^{(k+1)}}(x)\overline{\psi}_{a^{(k+1)}}(y) \\
&+ \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} M_r \left(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r) \right) \psi_{a^{(r+1)}}(x)\overline{\psi}_{a^{(r+1)}}(y) =: J_1 + J_2,
\end{aligned}$$

where $x \in G$, $y \in I_r(x) \setminus I_{r+1}(x)$.

$$J_1 = \sum_{a_0=0}^{m_0-1} \cdots \sum_{a_{r-1}=0}^{m_{r-1}-1} \sum_{a_{r+1}=0}^{m_{r+1}-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \left(\sum_{a_r=0}^{m_r-1} \varphi_r^{a_r}(x_r)\overline{\varphi}_r^{a_r}(y_r)\phi(x,y) \right),$$

where $\phi(x,y)$ is not depend on a_r. By the (2.2) equality which is in the proof of Paley lemma, it is valid that $J_1 = 0$.

$$J_2 = M_r \sum_{a_0=0}^{m_0-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \left(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r) \right) \psi_{a^{(r+1)}}(x)\overline{\psi}_{a^{(r+1)}}(y).$$

It is clear that J_2 is not depend on $a_1, a_2, \dots, a_{r-1}$, for this reason

$$J_2 = M_r^2 \sum_{a_r=0}^{m_r-1} \left(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r) \right) \sum_{a_{r+1}=0}^{m_{r+1}-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \psi_{a^{(l)}}(x)\overline{\psi}_{a^{(l)}}(y)$$
$$= M_r^2 \sum_{a_r=0}^{m_r-1} \left(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r) \right)$$
$$\times \prod_{l=r+1}^{s-1} \left(\sum_{a_l=0}^{m_l-1} \varphi_l^{a_l}(x_l)\overline{\varphi}_l^{a_l}(y_l) \right) \psi_{n^{(s)}}(x)\overline{\psi}_{n^{(s)}}(y).$$

By (2.2) we can state that J_2 is 0 if $x_l \neq y_l$ for any $r < l < s$. Since G is a bounded group, if $x_l = y_l$,$r < l < s$ we have

$$|K_{n^{(s)},M_s}(x,y)| \leq cM_rM_sd_{n^{(s)}}.$$

Then denoting by

$$A := \{y \in G : y_0 = x_0, ..., y_{r-1} = x_{r-1}, y_r \neq x_r,$$
$$, y_{r+1} = x_{r+1}, ..., y_{s-1} = x_{s-1}\},$$

we have

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_{n^{(s)},M_s}(x,y)|d\mu(y) \leq \int_A cM_rM_sd_{n^{(s)}}d\mu(y) =$$
$$= cM_rM_sd_{n^{(s)}}\frac{m_r-1}{M_s} \leq cM_rd_{n^{(s)}}.$$

Since $n \geq M_{|n|}$, by (2.21) and (2.22)

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|d\mu(y) < \frac{c}{M_{|n|}} \sum_{s=0}^{r} M_sd_{n^{(s)}} + \frac{c}{M_{|n|}} \sum_{s=r+1}^{|n|} M_rd_{n^{(s)}} \leq$$
$$\leq \frac{c}{M_{|n|}} M_rd_{n^{(r)}} + \frac{c}{M_{|n|}} M_rd_{n^{(r)}}(|n|-r).$$

Then

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|d\mu(y) < \frac{c}{M_{|n|}} M_r d_{n^{(r)}}(|n|-r+1).$$

G is a disjoint union of sets $I_r(x) \setminus I_{r+1}(x)$, $r \in \mathbf{N}$, where x is a fix element of G. If $r > |n|$, the $|K_n(x,y)|$ kernels depend only on x if $y \in I_r(x)$. For this reason if $x \in G$, then we get that

$$|K_n(x,x)| \leq \frac{1}{n} \sum_{l=0}^{n-1} \sum_{k=0}^{|n|} M_k c d_{n^{(k)}} < C M_{|n|},$$

and hence

$$\begin{aligned} \sum_{r=|n|+1}^{\infty} \int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|d\mu(y) &= \int_{I_{|n|+1}(x)} |K_n(x,y)|\, d\mu(y) \\ &= \frac{|K_n(x,x)|}{M_{|n|}} \leq C. \end{aligned}$$

Since $\dfrac{d_j^{(n_j)}}{m_j} < \dfrac{1}{\sqrt{2}}$ it follows that

$$\begin{aligned} \int_G |K_n(x,y)|d\mu(y) &= \sum_{r=0}^{\infty} \int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|d\mu(y) \\ &\leq \sum_{r=0}^{|n|} \frac{C}{M_{|n|}} M_r d_{n^{(r)}}(|n|-r+1) + C \\ &\leq \frac{C}{M_{|n|}} d_{|n|}^{(n_{|n|})} \sum_{r=0}^{|n|} \frac{M_{|n|}}{m_r \dots m_{|n|-1}} d_r^{(n_r)} \dots d_{|n|-1}^{(n_{|n|-1})}(|n|-r+1) + C \\ &\leq c \sum_{k=0}^{\infty} \frac{k+1}{(\sqrt{2})^k} + C \end{aligned}$$

for each $x \in G$, where $m_r...m_{|n|-1} = 1$ if $r = |n|$. Since the above series is convergent, for each $x \in G$ there exists a C positive constant such that

$$\int_G |K_n(x,y)|d\mu(y) \leq C.$$

This completes the proof of the lemma. □

In analogous way we can prove that there is C positive constant such that

$$\sup_{y\in G} \int_{G_m} |K_n(x,y)|d\mu(x) \leq C.$$

From above lemma we can get

Theorem 2.3.2. *If G is a bounded group and $f \in L^p(G)$, $1 \leq p \leq \infty$, then $\sigma_n f \to f$ in L^p-norm.*

Proof. It is sufficient to prove that σ_n operators are uniformly of type (p,p) when $1 \leq p \leq \infty$, since the $\sigma_n f \to f$ convergence is valid for each representative functions and for this reason we can apply the theorem of Banach-Steinhauss. From the interpolation theorem of Marczinkiewicz [24], it is sufficient to prove that σ_n operators are uniformly of type $(1,1)$ and (∞,∞). From Lemma 2.3.1, using the theorem of Fubini, if $f \in L^1(G)$

$$\|\sigma_n f\|_1 \leq \int_G \int_G |f(y)||K_n(x,y)|d\mu(y)d\mu(x) =$$
$$= \int_G |f(y)| \int_G |K_n(x,y)|d\mu(x)d\mu(y) \leq C\|f\|_1.$$

Then σ_n operators are uniformly of type $(1,1)$. If $f \in L^\infty(G)$

$$\|\sigma_n f\|_\infty \leq \|f\|_\infty \int_G |K_n(.,y)|d\mu(y)\|_\infty \leq C\|f\|_\infty.$$

Then σ_n operators are uniformly of type (∞,∞). This completes the proof of the theorem. □

Finally, we remark that Gát in [9] proved the pointwise convergence $\sigma_n f \to f$ a.e. ($f \in L^1(G)$). For the the Walsh case this proved by Fine [4], and for bounded (Abelian) Vilenkin groups proved by Simon and Pál [28]. The two-dimensional (Walsh) case $\sigma_{m,n} f \to f$ a.e. is discussed by Shipp, Móricz and Wade [16] ($\min(m, n) \to \infty$, $f \in H^\#$ (which is a certain "hybrid" Hardy space)), and by Gát [9] ($m, n \to \infty$), provided the integral lattice points (m, n) remain in some positive cone, $f \in L^1$).

Chapter 3

Fourier coefficients and absolute convergence

Before we see our statements in this chapter, we should remark that if the group G is not Abelian group then the system ψ is not bounded. This fact is important because the norm of the operators

$$T_n : L^1(G) \to \mathbf{C}, \qquad T_n f := \int_G f\overline{\psi_n}\, d\mu$$

is $\|\psi_n\|_\infty$. For this reason if the group G is not Abelian then there is a $f \in L^1(G)$ such that $\widehat{f}(n) \nrightarrow 0$, so the well known Riemann-Lebesgue lemma is not valid for non-commutative cases. In Section 3.1 we estimates of the Fourier coefficients of a function in $L^1(G)$ using it's modulus of continuity (see (2.11) and (2.12)). We should not be surprised that $\|\psi_n\|_\infty$ appears in the estimation. We give the concept of function with bounded fluctuation which was introduced by Onneweer and Waterman [18].

Benke in [2] was proved that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials, and that functions which are easily approximated by representative functions have absolutely convergent Fourier series. In Section 3.2, we study the absolute convergence of Fourier series based on the system of characters of G for functions which are constant on the conjugacy classes of G.

All of the results in this chapter appeared in [11].

3.1 Estimates of the Fourier coefficients

In (2.12) we can observe that $\omega_n(f, I)$ is a measure of the oscillation of f on I. Thus we say that a f function is of *p-bounded fluctuation* for some $1 \leq p < \infty$ if

$$\sup_{n\in\mathbf{N}} \left(\sum_{k=0}^{M_n-1} |\omega(f, I_n(k^*))|^p \right)^{\frac{1}{p}} < \infty,$$

where $k^* = (k_0, k_1, \dots) \in G$. A function is said to be of *bounded fluctuation* if it is of 1-bounded fluctuation. In this case define the *total fluctuation* by

$$\mathcal{F}\ell(f) := \sup_{n\in\mathbf{N}} \left(\sum_{k=0}^{M_n-1} |\omega(f, I_n(k^*))| \right).$$

In order to prove the theorems of this section we first prove the following lemma. τ_h represent the left translation operator (see (1.6)).

Lemma 3.1.1. *Let $f \in L^1(G)$, $n,\ k \in \mathbf{N}$. If $n > M_k$ then there is a $h \in I_k$ such that*

$$|\widehat{\tau_h f}(n) - \widehat{f}(n)| \geq |\widehat{f}(n)|.$$

Proof. Denote by $s = \max\{j \in \mathbf{N} : n_j \neq 0\}$ and let p be an arbitrary element of G_s. Moreover let $h(p)$ be the element of G with expansion

$h(p) = (\underset{0}{e}, \underset{1}{e}, \dots, \underset{s-1}{e}, \underset{s}{p}, \underset{s+1}{e}, \dots)$. Then $h(p) \in I_s \subset I_k$. If $\varphi_s^{n_s}$ is a normalized coordinate function of the group G_s, then there exist a $\sigma \in \Sigma_s$, and $i, j \in \{1, ..., d_\sigma\}$ such that

$$\varphi_s^{n_s} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}.$$

Using the fact that the measure μ is both right and left translation invariant and $n < M_{s+1}$ we have

$$\begin{aligned}
\widehat{\tau_{h(p)} f}(n) &= \int_G f(x) \overline{\psi_n(x(h(p))^{-1})}\, d\mu(x) \\
&= \int_G f(x) \overline{\varphi_s(x_s p^{-1})} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) \\
&= \int_G f(x) \overline{\sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x_s p^{-1})} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) \\
&= \int_G f(x) \sqrt{d_\sigma} \sum_{r=1}^{d_\sigma} \overline{u_{i,r}^{(\sigma)}(x_s) u_{j,r}^{(\sigma)}(p)} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) \\
&= \sum_{r=1}^{d_\sigma} u_{j,r}^{(\sigma)}(p) \int_G f(x) \overline{\sqrt{d_\sigma} u_{i,r}^{(\sigma)}(x_s)} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x).
\end{aligned}$$

The coordinate functions form an orthonormal system:

$$\sum_{p \in G_s} u_{j,r}^{(\sigma)}(p) = m_s \int_{G_s} u_{j,r}^{(\sigma)}(x)\, d\mu_s(x) = 0 \quad \Longrightarrow \quad \sum_{p \in G_s} \widehat{\tau_{h(p)} f}(n) = 0.$$

On the other hand, $\widehat{\tau_{h(e)} f}(n) = \widehat{f}(n)$. Denote by $\langle ., . \rangle$ the inner product (for the complex numbers $a + b\imath, c + d\imath$ $\langle a + b\imath, c + d\imath \rangle = ac + bd$). Thus

$$\widehat{f}(n) + \sum_{\substack{p \in G_s \\ p \neq e}} \widehat{\tau_{h(p)} f}(n) = 0 \quad \Longrightarrow \quad |\widehat{f}(n)|^2 + \sum_{\substack{p \in G_s \\ p \neq e}} \langle \widehat{\tau_{h(p)} f}(n), \widehat{f}(n) \rangle = 0.$$

Then there exists a $p \in G_s$, $p \neq e$ such that

$$\langle \widehat{\tau_{h(p)} f}(n), \widehat{f}(n) \rangle < 0 \qquad \Longrightarrow \qquad |\widehat{\tau_{h(p)} f}(n) - \widehat{f}(n)| \geq |\widehat{f}(n)|.$$

This completes the proof of the lemma. □

Theorem 3.1.2. *Let $f \in L^1(G)$, $n,\ k \in \mathbf{N}$. If $n > M_k$ then*

$$|\widehat{f}(n)| < \omega_k(f) \|\psi_n\|_\infty.$$

Proof. Let h be an element of G with satisfies the conditions of above lemma. By linearity of $\widehat{}$ we see that

$$\begin{aligned} |\widehat{f}(n)| &\leq |\widehat{\tau_h f}(n) - \widehat{f}(n)| = |\widehat{\tau_h f - f}(n)| \\ &= |\int_G (\tau_h f(x) - f(x)) \overline{\psi_n(x)}\, d\mu(x)| \\ &\leq \|\tau_h f - f\|_1 \|\psi_n\|_\infty \\ &\leq \omega_k(f) \|\psi_n\|_\infty \end{aligned}$$

which was to be proved. □

Similarly, we prove the following statement

Theorem 3.1.3. *Denote by $n \in \mathbf{N}$ and $s = \max\{j \in \mathbf{N} : n_j \neq 0\}$. If f is of bounded fluctuation, then*

$$|\widehat{f}(n)| \leq \frac{\mathcal{F}\ell(f)}{M_s} \|\psi_n\|_\infty.$$

Proof. Let h be an element of G with satisfies the conditions of above lemma. By linearity of $\widehat{\ }$ we see that

$$
\begin{aligned}
|\widehat{f}(n)| &\le |\widehat{\tau_h f}(n) - \widehat{f}(n)| = |\widehat{\tau_h f - f}(n)| \\
&= |\int_G (\tau_h f(x) - f(x))\overline{\psi_n(x)}\, d\mu(x)| \\
&\le \int_G |\tau_h f(x) - f(x)|\, d\mu(x) \|\psi_n\|_\infty \\
&= \sum_{k=0}^{M_s-1} \int_{I_s(k^*)} |\tau_h f(x) - f(x)|\, d\mu(x) \|\psi_n\|_\infty \\
&\le \frac{1}{M_s} \sum_{k=0}^{M_s-1} |\omega(f, I_s(k^*))| \|\psi_n\|_\infty \le \frac{\mathcal{F}\ell(f)}{M_s} \|\psi_n\|_\infty,
\end{aligned}
$$

since the sets $I_s(k^*)$ $(0 \le k \le M_s - 1)$ are disjoints, cover the set G and $\mu(I_s(k^*)) = \dfrac{1}{M_s}$. This completes the theorem. □

3.2 Absolute convergence of functions in $\mathcal{L}^p(G)$

Denote by p_k the number of conjugacy classes of the finite groups G_k $(k \in \mathbf{N})$. With them we construct the sequence $P_{k+1} := p_k P_k$ $k \in \mathbf{N}$ $(P_0 := 1)$. Then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$, $n_k \in \mathbf{N}$. This allows one to say that the $(n_0, n_1, \dots)$ sequence is the expansion of n with respect to the sequence $(p_0, p_1, \dots)$.

In addition, denote by $\chi_k^0 = 1$, $\chi_k^1, \dots, \chi_k^{p_k-1}$ the characters of the representations of the group G_k and let d_k^j be the dimension of the representation corresponding to the character χ_k^j. Then we obtain the characters of G in the form

$$\chi_n = \prod_{k=0}^{\infty} \chi_k^{n_k} \qquad (n = \sum_{k=0}^{\infty} n_k P_k;\ k \in \mathbf{N}).$$

We restrict the space $L^p(G)$ for the functions that are constant on every conjugacy classes of G. We denote this new space by $\mathcal{L}^p(G)$. The system of characters $\chi = (\chi_n, n \in \mathbf{N})$ of a non-abelian group is not complete in $L^1(G)$, but it is orthonormal and complete in $\mathcal{L}^1(G)$.

We denote by $\mathcal{A}$ the set of functions which have absolutely convergent Fourier series based in the system of characters of G. The Lipschitz class of order α will be denoted by $\mathrm{Lip}(\alpha)$. It is a closed subspace of the continuous functions endowed with the norm

$$\|f\|_{\mathrm{Lip}(\alpha)} := \sup_k \left[\sup_{x \in I_k} \|f(x\cdot) - f(\cdot)\|_\infty M_k^\alpha \right] + \|f\|_\infty.$$

The following two lemmas are used in the proof of the theorems bellow.

Lemma 3.2.1. *Let $f : G_i \to \mathbf{C}$, $j \in \{0, 1, \ldots, p_i - 1\}$, $i \in \mathbf{N}$. Thus there is a $h \in G_i$ such that (if $\chi_i^j \not\equiv 1$)*

$$\left| \sum_{x \in G_i} f(xh) \overline{\chi_i^j}(x) - \sum_{x \in G_i} f(x) \overline{\chi_i^j}(x) \right| \geq \left| \sum_{x \in G_i} f(x) \overline{\chi_i^j}(x) \right|.$$

Proof. Let $x \in G_i$. For simplicity we assume that the complex number $A := -\sum_{x \in G_i} f(x)\overline{\chi_i^j}(x)$ is on the first quadrant of the complex plane. If the complex number $B(h) := \sum_{x \in G_i} f(xh)\overline{\chi_i^j}(x)$ is also on the first quadrant for some $h \in G_i$, then our statement follows for this $h \in G_i$. If $B(h)$ is on the fourth quadrant of the complex plane for some $h \in G_i$, then we change h by h^{-1} and using the property $\chi_i^j(h^{-1}) = \overline{\chi_i^j(h)}$ we have that $B(h^{-1})$ is on the first quadrant, so we proved our statement for $h^{-1} \in G_i$. On the other hand,

$$\sum_{h\in G_i}\sum_{x\in G_i} f(xh)\overline{\chi_i^j}(x) = \sum_{h\in G_i}\sum_{x\in G_i} f(x)\overline{\chi_i^j}(xh^{-1})$$

$$= \sum_{h\in G_i}\sum_{x\in G_i} f(x)\sum_{r=1}^{d_i^j}\overline{u_{r,r}^\sigma}(xh^{-1}) = \sum_{h\in G_i}\sum_{x\in G_i} f(x)\sum_{r=1}^{d_i^j}\sum_{s=1}^{d_i^j}\overline{u_{r,s}^\sigma}(x)u_{s,r}^\sigma(h)$$

$$= \sum_{r,s=1}^{d_i^j}\left(\sum_{x\in G_i} f(x)\overline{u_{r,s}^\sigma}(x)\right)\sum_{h\in G_i} u_{s,r}^\sigma(h) = 0.$$

Thus we have that there is a $h \in G_i$ so that the corresponding complex number $B(h)$ is on the first or fourth quadrant of the complex plane. This completes the proof of the lemma. □

Lemma 3.2.2. *Let $f \in \mathcal{L}^1(G)$, $P_n \le k < P_{n+1}$ $(k, n \in \mathbf{N})$. Then there is a $h_n \in G_n$ and $h := h_n e_n = (e, e, \ldots, e, h_n, e, \ldots)$ such that*

$$|\widehat{\tau_h f}(k) - \widehat{f}(k)| \ge |\widehat{f}(k)|.$$

Proof. Let $x \in G$, $k_{(n-1)} := \sum_{i=0}^{n-1} k_i P_i$.

$$\overline{\chi_k}(xh^{-1}) = \prod_{i=0}^{n}\chi_i^{k_i}(xh^{-1}) = \left(\prod_{i=0}^{n-1}\chi_i^{k_i}(x)\right)\chi_n^{k_n}(xh^{-1})$$
$$= \chi_{k_{(n-1)}}(x)\chi_n^{k_n}(xh^{-1}).$$

Define $g : G_n \to \mathbf{C}$ by

$$g(x_n) := M_n^{-1}\sum_{x_0,\ldots,x_{n-1}}(E_{n+1}f)(x)\overline{\chi_{k_{(n-1)}}}(x) \qquad (x_n \in G_n).$$

Thus $\widehat{f}(k) = m_n^{-1} \sum_{x_n \in G_n} g(x_n)\overline{\chi}_n^{k_n}(x)$,

$$\widehat{\tau_h f}(k) = m_n^{-1} \sum_{x_n \in G_n} g(x_n)\overline{\chi}_n^{k_n}(xh^{-1}) = m_n^{-1} \sum_{x_n \in G_n} g(x_nh)\overline{\chi}_n^{k_n}(x).$$

Finally, Lemma 3.2.1 completes the proof of this lemma. □

Theorem 3.2.3. *Let* $\sup m < \infty$, $f \in \mathcal{L}^2(G)$. *If*

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{M_n - 1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}} < \infty, \qquad \textit{then} \qquad f \in \mathcal{A}.$$

Proof. Let $P_n \leq k < P_{n+1}$, $a := k_n$. Lemma 3.2.2 guaranties that there is a $h = (e, e, \dots, e, h_n, e, \dots)$ such that

$$|\widehat{\tau_h f}(k) - \widehat{f}(k)| \geq |\widehat{f}(k)|.$$

From the Cauchy's inequality we have

$$\begin{aligned}
\sum_{k=P_n}^{P_{n+1}-1} |\widehat{f}(k)| d_k &\leq \left(\sum_{k=P_n}^{P_{n+1}-1} (d_k)^2 \right)^{\frac{1}{2}} \left[\sum_{a=0}^{p_n - 1} \sum_{k=aP_n}^{(a+1)P_n - 1} |\widehat{f}(k)|^2 \right]^{\frac{1}{2}} \\
&\leq \left(\sum_{k=P_n}^{P_{n+1}-1} (d_k)^2 \right)^{\frac{1}{2}} \left[\sum_{a=0}^{p_n - 1} \sum_{k=aP_n}^{(a+1)P_n - 1} |\widehat{\tau_{h(a)} f}(k) - \widehat{f}(k)|^2 \right]^{\frac{1}{2}} \\
&\leq \sqrt{M_{n+1}} \sqrt{\sum_{a=0}^{p_n - 1} \|\tau_{h(a)} f - f\|_2^2},
\end{aligned}$$

since $d_k = \prod_{i=0}^{n} d_i^{k_i}$, $\sum_{k_i=0}^{p_i - 1} (d_i^{k_i})^2 = m_i \quad (i \in \mathbf{N})$. χ is an orthonormal system, then we can use the Bessel's inequality. On the other hand,

$$\begin{aligned} M_{n+1}\|\tau_{h(a)}f - f\|_2^2 &= M_{n+1}\sum_{k=0}^{M_n-1}\int_{I_n(k^*)}|f(xh)-f(x)|^2\,dx \\ &\leq m_n\sum_{k=0}^{M_n-1}|\omega^{(2)}(f,I_n(k^*))|^2. \end{aligned}$$

Since $p_n \leq m_n$ (and sequence m is bounded), then

$$\sum_{k=P_n}^{P_{n+1}-1}|\widehat{f}(k)|d_k \leq m_n\left(\sum_{k=0}^{M_n-1}|\omega^{(2)}(f,I_n(k^*))|^2\right)^{\frac{1}{2}}.$$

Thus

$$\|f\|_A := \sum_{k=0}^{\infty}|\widehat{f}(k)|d_k \leq m_n\sum_{n=0}^{\infty}\left(\sum_{\substack{t_i\in G_i\\ i<n}}|\omega^{(2)}(f,I_n(t))|^2\right)^{\frac{1}{2}} < \infty.$$

This completes the proof of the theorem. □

From the proof of the theorem we obtain that if $f \in \mathcal{L}^2(G)$ and

$$\sum_{n=0}^{\infty}m_n\left(\sum_{k=0}^{M_n-1}|\omega^{(2)}(f,I_n(k^*))|^2\right)^{\frac{1}{2}} < \infty, \qquad \text{then} \qquad f \in \mathcal{A}$$

independently of the fact that m is bounded or not.

The following statement is the generalization of a similar statement appeared in [24].

Theorem 3.2.4. *Let $f : G \to \mathbf{C}$ a continuous function that is constant in the conjugacy classes of G and suppose that exists a $1 \leq p \leq 2$ such that*

$$\sum_{n=0}^{\infty} \left(\sum_{\substack{t_i \in G_i \\ i<n}} |\omega(f, I_n(t))|^p \right)^{\frac{1}{p}} < \infty. \qquad \textit{Then} \qquad f \in \mathcal{A}.$$

Proof. Since $f \in \mathcal{L}^2(G)$ we have

$$\begin{aligned} \omega^{(2)}(f, I_n(t)) &= \sup_{h \in I_n} \left[M_n \int_{I_n(t)} |f(x+h) - f(x)|^2 \, dx \right]^{\frac{1}{2}} \\ &\leq \sup_{\substack{h \in I_n \\ x \in I_n(t)}} |f(x+h) - f(x)| = \omega(f, I_n(t)). \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{M_n - 1} |\omega^{(2)}(f, I_n(k^*))|^p \right)^{\frac{1}{p}} < \infty.$$

Using the inequality $\left(\sum_{i=1}^{N} |a_i|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{N} |a_i|^p \right)^{\frac{1}{p}}$ $(1 \leq p \leq 2)$ we have

$$\sum_{n=0}^{\infty} \left(\sum_{\substack{t_i \in G_i \\ i<n}} |\omega^{(2)}(f, I_n(t))|^2 \right)^{\frac{1}{2}} < \infty.$$

That is, the the conditions of Theorem 3.2.3 is fulfilled. This completes the proof of this theorem. $\square$

Corollary 3.2.5. *Let $f : G \to \mathbf{C}$ a continuous function that is constant in the conjugacy classes of G and suppose that*

$$\sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) < \infty \qquad (\sup m < \infty).$$

Then $f \in \mathcal{A}$.

Proof. The corollary is a consequence of the theorem since

$$\omega(f, I_n(t)) \leq \omega_n(f) \qquad (t \in G).$$

For this reason

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{M_n - 1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}} < \sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) < \infty.$$

This completes the proof of this corollary. □

Corollary 3.2.6. *Let $f \in Lip(\alpha)$ for some $\alpha > \frac{1}{2}$ $(\sup m < \infty)$. Then $f \in \mathcal{A}$.*

Proof. $\omega_n(f) \leq c M_n^{-\alpha}$ $(n \in \mathbf{N})$, thus

$$\sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) \leq c \sum_{n=0}^{\infty} M_n^{\frac{1}{2}-\alpha} < \infty.$$

Thus the conditions of the previous corollary is satisfied. □

Chapter 4

On Hardy-norm of operators with property Δ

The property Δ of operators was introduced by Schipp in [21] and he proved that some boundedness property with respect to L^p-norms of this operators are inherited by the sum of them. In Theorem 4.1.1 we resume these results. Several operators occurring in the theory of martingales can be given in this form. Renowned examples are the martingale-transforms that obviously are of this form, but more complicated sums of operators having the property Δ will be also considered in this chapter, i.e., the conjugate martingale transforms. Another interesting example is given by Gát in [8], [7] and [6] and by the author of this work in [29], [30] and [31] named $\psi\alpha$ or Vilenkin-like systems. This chapter shows the results appeared in [32].

In Section 1 we introduce some concepts and resume the results of [21]. Furthermore, we prove a lemma which we often use throughout this chapter and state the theorem of Hausdorff-Young for finite groups. The extension of Theorem 4.1.1 to the Hardy and BMO-norms are given in Section 4.2.

Schipp applied Theorem [21] to prove the significant result that the

Fourier-Vilenkin expansions of the function $f \in L^p$ converge to f in L^p-norm ($1 < p < \infty$). Similarly, in Section 4.3 we use our statements to show that the conjugate martingale transforms with matrix operators acting on the generalized Rademacher series of Vilenkin groups are bounded on L^r, H_r^s, BMO_r and H_1^s for $r = p, q$ ($p \geq 2, 1/p + 1/q = 1$) in so far as these matrices are uniformly of type (ℓ^q, ℓ^p) and uniformly bounded on ℓ^2. These transforms was first introduced by Gundy [12] and the above results was proved by Weisz [37] for bounded Vilenkin groups.

4.1 Preliminaries

Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space and $(\mathcal{A}_n, n \in \mathbf{N})$ be a sequence of σ-algebras for which

$$\{\emptyset, \Omega\} = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{A}$$

and $\mathcal{A} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$. Let $\mathcal{B} \subseteq \mathcal{A}$ be a σ-algebra. Denote by $L^p(\mathcal{B})$ the complex Lebesgue space $L^p(\Omega, \mathcal{B}, \mu)$ for $1 \leq p \leq \infty$, $L^p := L^p(\mathcal{A})$, L^0 the set of $\mathcal{A}$ measurable step functions and $L_0^p := \{f \in L^p : Ef = 0\}$, where Ef is the mean value of the complex function f.

The set $L \subseteq L^1$ is say to be a $\mathcal{B}$-*linear subspace*, if for every function $f, g \in L$ and $\lambda_1, \lambda_2 \in L^\infty(\mathcal{B})$, we have $\lambda_1 f + \lambda_2 g \in L$. A mapping $T : L \to L^1$ defined on the $\mathcal{B}$-linear subspace $L \subseteq L^1$ will be called $\mathcal{B}$-*linear*, if for any $f, g \in L$ and $\lambda_1, \lambda_2 \in L^\infty(\mathcal{B})$ we have $T(\lambda_1 f + \lambda_2 g) = \lambda_1 Tf + \lambda_2 Tg$.

The conditional expectation operator of the function $f \in L^1$ relative to the σ-algebra $\mathcal{B}$ will be denoted by $E(f|\mathcal{B})$, furthermore $E_n f := E(f|\mathcal{A}_n)$ and $Ef = E_0 f$. We say that the mapping $T : L^p \to L^q$, $1 \leq p, q < \infty$ *has type* $(\mathcal{A}_n, p, q)$ if there exists a $C > 0$ such that for all $f \in L^p$

$$(E_n|Tf|^q)^{\frac{1}{q}} \le C\,(E_n|f|^p)^{\frac{1}{p}}\,. \tag{4.1}$$

In the same way let X, Y be two normed spaces with $\|.\|_X$ and $\|.\|_Y$ norms, respectively. The operator $T : X \to Y$ is *of type* (X, Y) if there is a $C > 0$ such that

$$\|Tf\|_Y \le C\|f\|_X \qquad (f \in X).$$

If $X = Y$ we only say that T is *bounded on* X. In case $X = L^p$ and $Y = L^q$ we will say that T is of type (p, q).

The operator $T : L^1 \to L^1$ is $\mathcal{B}$-*selfadjoint* if for every $f, g \in L^1$

$$E((Tf)\overline{g}|\mathcal{B}) = E(f\overline{Tg}|\mathcal{B}). \tag{4.2}$$

In this chapter we often use the concept of dual space. If $X, Y \subseteq L^1$ be two normed spaces with $\|.\|_X$ and $\|.\|_Y$ norms, respectively, then the fact that the *dual space* of X is Y $(X' = Y)$ consists of two parts namely

(a) an inequality:

$$|E(f\overline{\varphi})| \le c_1\|f\|_X\|\varphi\|_Y \qquad (f \in X, \varphi \in Y),$$

where

$$E(f\overline{\varphi}) := \lim_{n\to\infty} E(E_n f\overline{E_n\varphi}).$$

(b) every linear bounded functional L on X is of the form

$$L(f) = E(f\overline{\varphi}),$$

where $\varphi \in Y$ and $\|\varphi\|_Y \le c_2\|L\|$.

Further, we introduce the following notations relative to the martingale $(E_n f,\ n \in \mathbf{N},\ f \in L_0^1)$.

$$\begin{aligned} \mathit{\Delta}_n f &:= E_n f - E_{n-1} f \qquad (\mathit{\Delta}_0 := 0), \\ f^* &:= \sup_{n \in \mathbf{N}} |E_n f|, \\ S(f) &:= \left(\sum_{n=1}^{\infty} |\mathit{\Delta}_n f|^2 \right)^{\frac{1}{2}}, \\ s(f) &:= \left(\sum_{n=1}^{\infty} E_{n-1} |\mathit{\Delta}_n f|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

It is clear that

$$E_n \circ E_m = E_{\min(n,m)}, \quad \mathit{\Delta}_n \circ \mathit{\Delta}_m = \delta_{mn} \mathit{\Delta}_n \quad (n, m \in \mathbf{N}), \tag{4.3}$$

where δ_{mn} is the Kronecker symbol.

We say that the sequence of operators $(T_n, n \in \mathbf{P})$ satisfy the *condition $\mathit{\Delta}$* if

$$T_n \circ \mathit{\Delta}_n = \mathit{\Delta}_n \circ T_n = T_n. \tag{$\mathit{\Delta}$}$$

From (4.3) we can show that if the condition $\mathit{\Delta}$ is satisfied then

$$\begin{aligned} T_n \circ E_n = E_n \circ T_n = T_n, \\ E_{n-1} \circ T_n = T_n \circ E_{n-1} = 0 \end{aligned} \qquad (n \in \mathbf{P}). \tag{4.4}$$

F. Schipp [21] discovered that some boundedness properties of operators T_n are inherited by the operator

$$Tf := \sum_{n=1}^{\infty} T_n f.$$

Theorem 4.1.1 (F. Schipp [21]).

(i) *Let $1 < p < \infty$, and let T_n be a $\mathcal{A}_{n-1}$-selfadjoint linear operators with property Δ ($n \in \mathbf{P}$). If the operator T is of type (p, p), then it is also of type (q, q), where $1/p + 1/q = 1$.*

(ii) *Let $(T_n, n \in \mathbf{P})$ be a sequence of linear operators with property Δ and let $p \geq 2$. If the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, p, p)$ then the operator T is of type (p, p).*

(iii) *Let $(T_n, n \in \mathbf{P})$ be a sequence of $\mathcal{A}_{n-1}$-linear operators having property Δ. If T_n is uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, 1, 1)$ at the same time, then T is of type $(2, 2)$ and of weak type $(1, 1)$, i.e., there is a positive constant c such that for every number $y > 0$ and every function $f \in L^1$*

$$\mu\{|Tf| > y\} \leq c\|f\|_1/y.$$

In Section 4.2 we prove the same statements on martingale Hardy and BMO spaces. In this regard we prove the following lemma.

Lemma 4.1.2. *Let $X, Y \subseteq L^1$ be two Banach spaces wherein the set of step function L^0 is dense and $X' = Y$. Suppose that $(T_n, n \in \mathbf{P})$ is a sequence of $\mathcal{A}_{n-1}$-selfadjoint linear operators with property Δ. If the operators $\sum_{n=1}^{N} T_n$ are uniformly bounded on X, then T is also bounded on Y.*

Proof. First we prove that there is an absolute constant $C > 0$ such that for every $N \in \mathbf{P}$

$$\left\| \sum_{n=1}^{N} T_n \varphi \right\|_Y \leq C \|\varphi\|_Y \qquad (\varphi \in L^0).$$

Thus we obtain our statement as a consequence of the theorem of Banach-Steinhaus.

By the concept of dual space we can show that

$$\|\varphi\|_Y \le c_2 \sup\{|E(f\overline{\varphi})| : \|f\|_X = 1\} \qquad (\varphi \in Y). \tag{4.5}$$

Let $k, N \in \mathbf{P}$, $f \in X$, $\|f\|_X = 1$, $\varphi \in Y_0$ and denote $G_N := \sum_{n=1}^N T_n$. Then by (3) and (4) we have

$$E(E_k f \overline{E_k G_N \varphi}) = \sum_{n=1}^N E(E_k f \overline{T_n E_k \varphi}).$$

Using that T_n is $\mathcal{A}_{n-1}$-selfadjoint, we obtain that

$$E(E_k f \overline{T_n E_k \varphi}) = E(E_k T_n f \overline{E_k \varphi}),$$

so by the inequality in the concept of duality

$$\begin{aligned} \left|E(f\overline{G_N\varphi})\right| = \lim_{k\to\infty} \left|E(E_k f \overline{E_k G_N \varphi})\right| = \lim_{k\to\infty} \left|E(E_k G_N f \overline{E_k\varphi})\right| \le \\ \le c_1 \|G_N f\|_X \|\varphi\|_Y \le c\|f\|_X\|\varphi\|_Y = c\|\varphi\|_Y \end{aligned}$$

because the operators G_N are uniformly bounded on X. Since $G_N\varphi \in Y$, by (4.5) we have

$$\|G_N\varphi\|_Y \le c_2 \sup\{\left|E(f\overline{G_N\varphi})\right| : \|f\|_X = 1\},$$

from which our statement follows. □

In Section 4.3 we use the Hausdorff-Young inequality for cyclic groups $\mathcal{Z}_m := \{0, 1, ..., m-1\}$ with measure μ such that the measure of a singleton is $1/m$. It's character system

$$r^k(x) := \exp(2\pi i k x/m) \qquad (0 \le k < m)$$

is the generalized Rademacher system. For every $f : \mathcal{Z}_m \to \mathbf{C}$ denote by $\hat{f} : \mathcal{Z}_m \to \mathbf{C}$, the *Fourier transform* of f, that is

$$\hat{f}(k) := \int_{\mathcal{Z}_m} f\overline{r^k} d\mu \qquad (0 \leq k < m),$$

and set

$$\|f\|_p := \left(\frac{1}{m}\sum_{k=0}^{m-1} |f(k)|^p\right)^{\frac{1}{p}}, \qquad \|f\|_{\ell^p} := \left(\sum_{k=0}^{m-1} |f(k)|^p\right)^{\frac{1}{p}}.$$

.

Theorem 4.1.3 (Hausdorff-Young). *Let* $1 \leq q \leq 2$ *and* $1/p+1/q = 1$, *then for every function* $f : \mathcal{Z}_m \to \mathbf{C}$ *we have*

(i) $$\|\hat{f}\|_{\ell^p} \leq \|f\|_q,$$

and

(ii) $$\|f\|_p \leq \|\hat{f}\|_{\ell^q}.$$

4.2 Martingale Hardy and BMO spaces

Throughout this work we use the notations employed by F. Weisz in [38], and $C > 0$ will denote an absolute constant which will not necessarily be the same at different occurrences. For $1 \leq p < \infty$ we shall consider the following martingale Hardy spaces:

$$\begin{aligned} H_p^s &:= \{f \in L_0^1 : \|f\|_{H_p^s} := \|s(f)\|_p < \infty\}, \\ H_p^S &:= \{f \in L_0^1 : \|f\|_{H_p^S} := \|S(f)\|_p < \infty\}, \\ H_p^* &:= \{f \in L_0^1 : \|f\|_{H_p^*} := \|f^*\|_p < \infty\}, \end{aligned}$$

and BMO spaces

$$BMO_p^- := \{\varphi \in L_0^p : \|\varphi\|_{BMO_p^-} := \sup_{k\geq 1} \left\| (E_k|\varphi - E_{k-1}\varphi|^p)^{\frac{1}{p}} \right\|_\infty < \infty\},$$
$$BMO_p := \{\varphi \in L_0^p : \|\varphi\|_{BMO_p} := \sup_{k\geq 1} \left\| (E_k|\varphi - E_k\varphi|^p)^{\frac{1}{p}} \right\|_\infty < \infty\}.$$

If every σ-algebra $\mathcal{A}_n$ are generated by finitely many (set) atoms (e.g m is bounded) then we define the VMO *spaces* by

$$VMO_p^- := \{\varphi \in BMO_p^- : \lim_{k\to\infty} \left\| (E_k|\varphi - E_{k-1}\varphi|^p)^{\frac{1}{p}} \right\|_\infty = 0\},$$
$$VMO_p := \{\varphi \in BMO_p : \lim_{k\to\infty} \left\| (E_k|\varphi - E_k\varphi|^p)^{\frac{1}{p}} \right\|_\infty = 0\}.$$

Before we state our results it is necessary to remark that the series $\sum_{n=1}^{\infty} T_n$ is finite in the set $L = \bigcup_{n\in\mathbf{P}} L^p(\mathcal{A}_n)$ or L^0 (L and L^0 are everywhere dense in L^p since the σ-algebras $\mathcal{A}_n$ generate $\mathcal{A}$ [12]). Thus in the case that the operators T_n are linear, from the theorem of Banach-Steinhaus, if the series $\sum_{n=1}^{N} T_n$ ($N \in \mathbf{P}$) are uniformly bounded on the Banach space X, then the operator T is bounded on X. The normed spaces that appear in this work are Banach spaces in which the set L^0 is dense.

On the other hand, if the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$, then the stochastic sequence $(\sum_{k=1}^{n} T_k f, \mathcal{A}_n)_{n\geq 1}$ is a uniformly integrable martingale for $f \in L^2$. This follows from the inequality

$$\sup_N E\left|\sum_{k=1}^{N} T_k f\right|^2 < \infty \qquad (f \in L^2).$$

which was proved in [14]. In this case the series $\sum_{k=1}^{\infty} T_k f$ is convergent with probability 1.

In addition if $T_n(L^p) \subseteq L^p$ $(n \in \mathbf{P})$ for any $p \geq 2$, then T is bounded on L^p that is the linearity of T_n is not required in the point (ii) of Theorem 4.1.1.

First we extend Theorem 4.1.1 to the spaces H_p^s and BMO_p.

Theorem 4.2.1. *Let T_n $(n \in \mathbf{P})$ be operators with property Δ and uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$. Then the operator T is bounded on H_p^s $(p \geq 1)$.*

Proof. Let $p \geq 1$ and $f \in H_p^s$. The stochastic sequence

$$\left(\sum_{k=1}^{n} T_k f, \mathcal{A}_n\right)_{n \geq 1}$$

is a martingale, therefore we have

$$\left\|\sum_{n=1}^{\infty} T_n f\right\|_{H_p^s} = \left\|\left(\sum_{k=1}^{\infty} E_{k-1}|T_k \Delta_k f|^2\right)^{\frac{1}{2}}\right\|_p .$$

Since the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$, there is a positive constant c such that

$$E_{k-1}|T_k f|^2 \leq c E_{k-1}|f|^2 \qquad (k \in \mathbf{P}).$$

Consequently,

$$\left\|\sum_{n=1}^{\infty} T_n f\right\|_{H_p^s} \leq c \left\|\left(\sum_{k=1}^{\infty} E_{k-1}|\Delta_k f|^2\right)^{\frac{1}{2}}\right\|_p = c\|f\|_{H_p^s},$$

from which our statement directly follows. □

Theorem 4.2.2. *Let T_n $(n \in \mathbf{P})$ be operators with property Δ and uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, p, p)$ at the same time for some $p \geq 2$. Then it is valid that*

(i) The operator T is bounded on BMO_p.

(ii) If the operators T_n are linear and $\mathcal{A}_{n-1}$-selfadjoint then T is also bounded on BMO_q, where $1/p + 1/q = 1$.

Proof. Let $\varphi \in BMO_p$. Then

$$\left\| \sum_{n=1}^{\infty} T_n \varphi \right\|_{BMO_p} = \sup_{k \geq 1} \left\| \left(E_k \left| \sum_{n=k+1}^{\infty} T_n \varphi \right|^p \right)^{\frac{1}{p}} \right\|_{\infty} . \tag{4.6}$$

Let $k \in \mathbf{P}$ and choose $A \in \mathcal{A}_k$. Then the operators $\chi_A T_n$ $(n > k)$ have property Δ and are uniformly of type $(\mathcal{A}_{n-1}, p, p)$, since

$$E_{n-1} |\chi_A T_n f|^p \leq E_{n-1} |T_n f|^p \leq c E_{n-1} |f|^p$$

(the constant c is not depend on the set A). Hence, from the point (ii) of Theorem 4.1.1 using that $BMO_p \subseteq L_0^p$ we have that for every $f \in L^p$ $(p \geq 2)$ and $A \in \mathcal{A}_k$

$$E \chi_A \left| \sum_{n=k+1}^{\infty} T_n \varphi \right|^p \leq c E \chi_A \left| \sum_{n=k+1}^{\infty} \Delta_n \varphi \right|^p , \tag{4.7}$$

from which we have

$$E_k \left| \sum_{n=k+1}^{\infty} T_n \varphi \right|^p \leq c E_k \left| \sum_{n=k+1}^{\infty} \Delta_n \varphi \right|^p \leq c E_k |\varphi - E_k \varphi|^p,$$

and from (4.2) we obtain

$$\left\| \sum_{n=1}^{\infty} T_n \varphi \right\|_{BMO_p} \leq c \sup_{k \geq 1} \left\| (E_k |\varphi - E_k \varphi|^p)^{\frac{1}{p}} \right\|_{\infty} = c \|\varphi\|_{BMO_p}$$

which was to be proved.

The proof of (ii) is based in the lemma for $X = L_0^p$, $Y = L_0^q$ and $T = \sum_{n=k+1}^{N} \chi_A T_n$ from which we obtain (4.7) for q in place of p and for finite sums. Then as before, we obtain our statements from the theorem of Banach-Steinhaus.

This completes the proof of the theorem. □

Using the interpolation argument for $BMO_p \subset L_0^p \subset H_p^s$ $(p \geq 2)$ we have

Corollary 4.2.3. *Let* $(T_n, n \in \mathbf{P})$ *be a sequence of linear operators with property* Δ*. If for any* $p \geq 2$ *the operators* T_n *are uniformly of type* $(\mathcal{A}_{n-1}, 2, 2)$ *and uniformly bounded on* BMO_p*, then the operator* T *is bounded on* BMO_p*,* L_0^p*,* BMO_q *and* L_0^q *where* $1/p + 1/q = 1$*.*

Similar statement is valid for BMO_p^- and H_p^S. In this regard we use the equivalence of the BMO_p^- spaces $(1 \leq p < \infty)$, hence denote everyone by BMO_2^-. Furthermore, we also used (see Garsia [5] and Weisz [38]) that the spaces H_p^S and L_0^p are equivalent for $p > 1$. For this reason we restrict our attention to H_1^S. Denote by BD_q $(1 \leq p \leq \infty)$ the spaces of functions f $(Ef = 0)$ such that

$$\|f\|_{BD_q} := \|\sup_n |\Delta_n f|\|_q < \infty.$$

Theorem 4.2.4. *Let* $(T_n, n \in \mathbf{P})$ *be a sequence of operators with property* Δ*. If the operators* T_n *are uniformly of type* $(\mathcal{A}_{n-1}, 2, 2)$ *and uniformly bounded on* BMO_2^- *then the operator* T *is also bounded on* BMO_2^-*. In addition if the operators* T_n *are linear,* $\mathcal{A}_{n-1}$*-selfadjoint and the* σ*-algebras* $\mathcal{A}_n$ *are generated by finitely many atoms, then* T *is also bounded on* H_1^S*.*

Proof. Using the inequalities in [38]

$$\sup\{\|\varphi\|_{BMO_2}, \|\varphi\|_{BD_\infty}\} \leq \|\varphi\|_{BMO_2^-}, \tag{4.8}$$

and

$$\|\varphi\|_{BMO_2^-} \le \|\varphi\|_{BMO_2} + \|\varphi\|_{BD_\infty}, \tag{4.9}$$

we obtain from the corollary that

$$\begin{aligned}\|T\varphi\|_{BMO_2^-} &\le \|T\varphi\|_{BMO_2} + \|T\varphi\|_{BD_\infty} \le c\|\varphi\|_{BMO_2} + \|T_n\varphi\|_{BD_\infty} \le \\ &\le c\|\varphi\|_{BMO_2^-} + \|T_n\varphi\|_{BMO_2^-} \le c\|\varphi\|_{BMO_2^-}\end{aligned}$$

which means that T is bounded on BMO_2^-.

F. Schipp [23] was proved that if the σ-algebras $\mathcal{A}_n$ are generated by finitely many atoms, then $(VMO_2^-)^* = H_1^S$. Thus the linearity of T_n guarantees that the conditions in the lemma are satisfy by $X = VMO_2^-$, $Y = H_1^S$.

This completes the proof of the theorem. □

Finally we prove a similar statement to the point (iii) of Theorem 4.1.1 using the Davis decomposition of martingale of H_1^S.

Theorem 4.2.5. *Let $(T_n, n \in \mathbf{P})$ be a sequence of linear operators with property Δ. If the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, 1, 1)$ at the same time, then T is bounded on H_1^S.*

Proof. First we introduce the following space

$$\mathcal{G}_p := \{f \in L_0^p : \|f\|_{\mathcal{G}_p} := \left\|\sum_{n=0}^{\infty} |\Delta_n f|\right\|_p < \infty\}.$$

It is easy to check that

$$\|f\|_{H_1^S} \le \|f\|_{\mathcal{G}_1} \qquad (f \in L_0^1).$$

Let $f \in H_1^S$. From Davis decomposition we have that there exist $g \in \mathcal{G}_1$ and $h \in H_1^s$ such that $f = h + g$ and

$$\|g\|_{\mathcal{G}_1} \le c\|f\|_{H_1^S}, \qquad \|h\|_{H_1^s} \le c\|f\|_{H_1^S}.$$

We can readily verify that if the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 1, 1)$ then T is bounded on $\mathcal{G}_1$. From Theorem 4.2.1 T is also bounded on H_1^s. Using the

$$\|f\|_{H_1^S} \leq c\|f\|_{H_1^s} \qquad (f \in L_0^1)$$

inequality we have

$$\begin{aligned}\|Tf\|_{H_1^S} = \|T(h+g)\|_{H_1^S} \leq \|Th\|_{H_1^s} + \|Tg\|_{\mathcal{G}_1} \leq \\ \leq c\|h\|_{H_1^s} + c\|g\|_{\mathcal{G}_1} \leq c\|f\|_{H_1^S}\end{aligned}$$

which was to be proved. □

4.3 Conjugate martingale transforms

In this section suppose G is a Vilenkin group (see Chapter 1). Denote by $\mathcal{A}_n$ the σ-algebra generated by the finite sets $I_n(x)$, $x \in G$ and $n \in \mathbf{N}$. Then $\mathcal{A}_0 \subset \mathcal{A}_1 \subset ... \subset \mathcal{A}$ and $\mathcal{A} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$, where $\mathcal{A}$ is the σ-algebra which contain the measurable sets. Set

$$S_n f := \sum_{k=0}^{n-1} E(f\overline{\psi_k})\psi_k.$$

It is easy to see that

$$E_{n-1}f(x) = S_{M_n}f(x) = M_n \int_{I_{n-1}(x)} f d\mu \qquad (x \in G_m, n \in \mathbf{P}), \tag{4.10}$$

and the martingale difference sequence is given by

$$\varDelta_n f := \sum_{k=1}^{m_n-1} E_{n-1}(f\overline{r_n^k})r_n^k. \tag{4.11}$$

The transform which first were introduced by Gundy [12] is given at follows. Let $A := (A_n, n \in \mathbf{P})$ be a sequence of complex matrices with $m_n - 1$ rows and columns, respectively. We assume that the Euclidean norms of A_n $(n \in \mathbf{P})$ are uniformly bounded, i.e. A_n are uniformly bounded on ℓ^2. Define the operator T_n by

$$T_n f := \sum_{k=1}^{m_n-1} (A_n v_n)^{(k)} r_n^k$$

where $v_n := (E_{n-1}(f\overline{r_n^k}))_{k=1}^{m_n-1}$. Then we say that the operator $T := \sum_{n=1}^{\infty} T_n$ is a *conjugate martingale transform*. If the A_n matrices are diagonal we obtain a *multiplier transform*. We should remark that P. Simon [26] used a concrete multiplier operator to prove the convergence of Fourier series in L^p-norm $(1 < p < \infty)$.

It is easy to see that $(T_n, n \in \mathbf{P})$ is a sequence of $\mathcal{A}_{n-1}$-selfadjoint linear operators with property Δ. From Theorem 4.2.1 and 4.2.2 we can state that

Theorem 4.3.1. *Let $p \geq 2$ and 1/p+1q=1. If the matrices A_n $(n \in \mathbf{P})$ are uniformly of type (ℓ^q, ℓ^p) and uniformly bounded on ℓ^2 then T is a bounded operator on L^r, H_r^s, BMO_r and H_1^s for r=p,q.*

Proof. It will be sufficient to show that the operator T_n are uniformly of type $(\mathcal{A}_{n-1}, p, p)$. In this regard note from (4.11) that $\Delta_n f$ in essential, is given by a Fourier sum defined in $\mathcal{Z}_{m_n}$. Hence it is sufficient to prove that for every cyclic group $\mathcal{Z}_m$ of order m, for every function $f : \mathcal{Z}_m \to \mathbf{C}$ with $\hat{f}(0) = 0$ and matrix $A : \mathbf{C}^{m-1} \to \mathbf{C}^{m-1}$ with Euclidean norm $\|A\|$ the operator

$$T_A f = \sum_{k=1}^{m-1} (A\hat{f})^{(k)} r^k.$$

satisfies the inequality

$$\|T_A f\|_p \leq \|A\|_{\ell^q, \ell^p} \|f\|_p.$$

Since the Fourier coefficients of $T_A f$ are 0 for $k = 0$ and $(A\hat{f})^{(k)}$ for $1 \leq k < m$, from the theorem of Hausdorff-Young we have

$$\begin{aligned} \|T_A f\|_p \leq \left(\sum_{k=1}^{m-1} \left| (A\hat{f})^{(k)} \right|^q \right)^{\frac{1}{q}} = \|(A\hat{f})\|_{\ell^q} \leq \\ \leq \|A\|_{\ell^q, \ell^p} \|\hat{f}\|_{\ell^p} \leq \|A\|_{\ell^q, \ell^p} \|f\|_q \leq \|A\|_{\ell^q, \ell^p} \|f\|_p \end{aligned}$$

which was to be proved. □

Summary

In this dissertation we discuss the convergence in norm of Fourier series and Fejér means with respect to a natural generalization on the Walsh-Paley and Vilenkin system. We take the complete product of arbitrary finite group (not necessarily commutative) and use the way given in the theory of harmonic analysis to introduce the orthonormal and complete systems with which we work (see [14]). These systems are called a representative product systems.

In Chapter 1 we study the structure of this groups and systems giving some examples in order to a better comprehension. Let G denote the compact group formed by the complete direct product of finite groups G_k with the product of the topologies, operations and measures (μ). Let $\{\varphi_k^s : 0 \leq s < |G_k|\}$ be the set of all normalized coordinate functions of the group G_k, and ψ be the product system of φ_k^s. The functions ψ_n ($n \in \mathbf{N}$) are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N}).$$

The sequence Ψ plays an important role in the convergence of Fourier series. At the end of this chapter we represent the group G and the different systems ψ_n on the interval $[0, 1]$ using Fine's map (see [35]).

Chapter 2 summarizes the results of [10]. We introduce the basic concepts of Fourier analysis and show the properties of the Dirichlet kernels to study the convergence in norm of Fourier series and Fejér

means. We also prove the Paley lemma for representative product systems and state it's consequences.

Lemma. *(Paley lemma) If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y) \\ 0 & \text{for } x \notin I_n(y) \end{cases},$$

where $I_n(y)$ is the interval

$$I_n(y) := \{x \in G : x_k = y_k, \text{ for } 0 \le k < n\} \quad (y \in G, n \in \mathbf{N}).$$

The partial sums of the Vilenkin-Fourier series of a function in $L^p(G)$ $(1 < p < \infty)$ converge in the appropriate norm to the function (Young [39], Schipp [21], Simon [26]). The statement above is not true for all cases if we take the complete product of arbitrary finite group (not necessarily commutative).

Theorem. *If G is a bounded group with unbounded sequence Ψ, then there is a $1 < p < 2$ for which the operator S_n is not of type (p, p).*

In this chapter (see also [34]) we prove that for an arbitrary group G there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm. We also introduce the concept of modulus of continuity to give class of functions for which the partial sums of it's Fourier series converge to the function in L^1 or in the uniform norm.

Theorem. *Let f be a function in $L^1(G)$ for which the following condition holds:*

$$\omega_k(f) = o(\Psi_k \sum_{j=0}^{k} m_j)^{-1}.$$

Then the sequence of partial sums $S_n f$ of Fourier series of f converges in L^1-norm to f.

Theorem. *Let f be a continuous function on G for which the following condition holds:*

$$\omega_k^{\infty}(f) = o(\Psi_k \sum_{j=0}^{k} m_k)^{-1}.$$

Then the sequence of partial sums $S_n f$ of Fourier series of f converges in uniform norm to f.

Theorem. *Let f be a continuous function on G for which the following condition holds:*

$$\sum_{k=0}^{\infty} m_k \omega_k^{\infty}(f) < \infty$$

and suppose the sequence Ψ is bounded. Then the sequence of partial sums $S_n f$ of Fourier series of f converges in uniform norm to f.

Finally, we obtain an important positive result.

Theorem. *If G is a bounded group, the Fejér means of a function $f \in L^p(G)$, $1 \leq p \leq \infty$ converge to the function in L^p-norm.*

In Chapter 3 we estimate the Fourier coefficients which not necessarily tend to zero, using the modulus of continuity of the function and the uniform norm of the system. This results were appeared in [11].

Theorem. *Let $f \in L^1(G)$, $n,\ k \in \mathbf{N}$. If $n > M_k$ then*

$$|\widehat{f}(n)| < \omega_k(f)\|\psi_n\|_{\infty}.$$

We specially study the functions with bounded fluctuation.

Theorem. *Denote by $n \in \mathbf{N}$ and $s = \max\{j \in \mathbf{N} : n_j \neq 0\}$. If f is of bounded fluctuation, then*

$$|\widehat{f}(n)| \leq \frac{\mathcal{F}\ell(f)}{M_s}\|\psi_n\|_{\infty}.$$

On the other hand, we consider an interesting class of functions, namely the ones that are constant on every conjugacy classes. The system of characters of the representations is complete in this class of functions, so we use characters to study the absolute convergence of series constructed in this way. We denote by $\mathcal{A}$ the set of functions which have absolutely convergent Fourier series based in the system of characters of G. The Lipschitz class of order α will be denoted by $\mathrm{Lip}(\alpha)$. Thus

Theorem. *Let* $\sup m < \infty$, $f \in \mathcal{L}^2(G)$. *If*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2\right)^{\frac{1}{2}} < \infty, \qquad \text{then} \qquad f \in \mathcal{A}.$$

Theorem. *Let* $f : G \to \mathbf{C}$ *a continuous function that is constant in the conjugacy classes of* G *and suppose that exists a* $1 \le p \le 2$ *such that*

$$\sum_{n=0}^{\infty}\left(\sum_{\substack{t_i \in G_i \\ i<n}} |\omega(f, I_n(t))|^p\right)^{\frac{1}{p}} < \infty. \qquad \text{Then} \qquad f \in \mathcal{A}.$$

Corollary. *Let* $f : G \to \mathbf{C}$ *a continuous function that is constant in the conjugacy classes of* G *and suppose that*

$$\sum_{n=0}^{\infty} \sqrt{M_n}\,\omega_n(f) < \infty \qquad (\sup m < \infty).$$

Then $f \in \mathcal{A}$.

Corollary. *Let* $f \in Lip(\alpha)$ *for some* $\alpha > \frac{1}{2}$ $(\sup m < \infty)$. *Then* $f \in \mathcal{A}$.

Chapter 4 treats the general case of product system adapting the results of Schipp [21] for the convergence in Hardy and BMO norms (see [32]). The property Δ of operators was introduced by Schipp in [21] and he proved that some boundedness property with respect to L^p-norms of this operators T_n are inherited by the sum T of them. For the different Hardy and BMO spaces we obtain

Theorem. *Let T_n $(n \in \mathbf{P})$ be operators with property Δ and uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$. Then the operator T is bounded on H_p^s $(p \geq 1)$.*

Theorem. *Let T_n $(n \in \mathbf{P})$ be operators with property Δ and uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, p, p)$ at the same time for some $p \geq 2$. Then it is valid that*

(i) The operator T is bounded on BMO_p.

(ii) If the operators T_n are linear and $\mathcal{A}_{n-1}$-selfadjoint then T is also bounded on BMO_q, where $1/p + 1/q = 1$.

Theorem. *Let $(T_n, n \in \mathbf{P})$ be a sequence of operators with property Δ. If the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and uniformly bounded on BMO_2^- then the operator T is also bounded on BMO_2^-. In addition if the operators T_n are linear, $\mathcal{A}_{n-1}$-selfadjoint and the σ-algebras $\mathcal{A}_n$ are generated by finitely many atoms, then T is also bounded on H_1^S.*

Theorem. *Let $(T_n, n \in \mathbf{P})$ be a sequence of linear operators with property Δ. If the operators T_n are uniformly of type $(\mathcal{A}_{n-1}, 2, 2)$ and $(\mathcal{A}_{n-1}, 1, 1)$ at the same time, then T is bounded on H_1^S.*

We use the convergence of operators with property Δ to study the conjugate martingale transforms defined on not necessarily bounded Vilenkin group. The transform which first were introduced by Gundy [12] is given at follows. Let $A := (A_n, n \in \mathbf{P})$ be a sequence of complex matrices with $m_n - 1$ rows and columns, respectively. We assume that the euclidean norms of A_n ($n \in \mathbf{P}$) are uniformly bounded, i.e. A_n are uniformly bounded on ℓ^2. Define the operator T_n by

$$T_n f := \sum_{k=1}^{m_n-1} (A_n v_n)^{(k)} r_n^k,$$

where $v_n := (E_{n-1}(f\overline{r_n^k}))_{k=1}^{m_n-1}$. Then we say that the operator $T := \sum_{n=1}^{\infty} T_n$ is a conjugate martingale transform. Weisz [37] studied these transforms for bounded Vilenkin groups. For this transforms we can state

Theorem. *Let $p \geq 2$ and $1/p+1q=1$. If the matrices A_n ($n \in \mathbf{P}$) are uniformly of type (ℓ^q, ℓ^p) and uniformly bounded on ℓ^2 then T is a bounded operator on L^r, H_r^s, BMO_r and H_1^s for r=p,q.*

Összefoglaló

Ebben az értekezésben a Fourier-sorok és a Fejér-közepek normakonvergenciával foglalkozunk, amelyeket a Walsh-Paley rendszer és a Vilenkin-rendszer egy természetes általánosításán értelmezünk. Tetszőleges véges (nem feltétlenül kommutatív) csoportok teljes direkt szorzatát vesszük és a harmonikus analízis elméletének segítségével bevezetjük azokat a teljes ortonormált rendszereket, amivel foglalkozni fogunk (lásd [14]). Ezeket a rendszereket reprezentatív szorzatrendszereknek nevezzük.

Az 1. fejezetben ezen csoportok és rendszerek struktúráját vizsgáljuk és néhány példát is megadunk a jobb megértés kedvéért. Jelölje G azt a kompakt csoportot, amely előáll G_k véges csoportok teljes direkt szorzataként és rendelkezik a véges csoportok szorzat-topológiáival, -műveleteivel és -mértékeivel. Legyen $\{\varphi_k^s : 0 \leq s < |G_k|\}$ a G_k csoport normalizált koordináta-függvényei és ψ a φ_k^s szorzatrendszere. A ψ_n $(n \in \mathbf{N})$ nem feltétlenül egyenletesen korlátosak, ezért definiáljuk:

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N}).$$

A Ψ sorozat fontos szerepet játszik a Fourier-sorok konvergenciájában. A fejezet végén reprezentáljuk a G csoportot és a különböző ψ rendszereket a $[0,1]$ intervallumon a Fine-leképezés segítségével (lásd [35]).

A 2. fejezet összegzi a [10]-es publikáció eredményeit. Bevezetjük a

Fourier-analízis alapfogalmait és megmutatjuk a Dirichlet-magok tulajdonságait, amelyekkel bizonyítjuk a Fourier-sorok és Fejér-közepek normakonvergenciával kapcsolatos állításokat. Bebizonyítjuk a Paley lemmát reprezentatív szorzatrendszerekre és ennek következményeit.

Lemma. *(Paley lemma) Ha $n \in \mathbf{N}$ és $x, y \in G$, akkor*

$$D_{M_n}(x, y) = \begin{cases} M_n & \text{ha } x \in I_n(y) \\ 0 & \text{ha } x \notin I_n(y) \end{cases},$$

ahol $I_n(y)$ a következő intervallum:

$$I_n(y) := \{x \in G : x_k = y_k, \text{ for } 0 \leq k < n\} \quad (y \in G, n \in \mathbf{N}).$$

Egy $L^p(G)$-beli függvény $(1 < p < \infty)$ Vilenkin-Fourier sora konvergál L^p normában a függvényhez. (Young [39], Schipp [21], Simon [26]). Az előző állítás nem igaz minden esetben, amikor vesszük tetszőleges véges (nem feltétlenül kommutatív) csoportok teljes direkt szorzatát.

Tétel. *Ha a G csoport korlátos de a Ψ sorozat nem korlátos, akkor van olyan $p > 1$ és olyan $f \in L^p(G)$ függvény, amelynek Fourier-sora nem konvergál a függvényhez L^p-normában.*

Ebben a fejezetben (lásd még [34]) bebizonyítjuk, hogy tetszőleges G csoport esetén van olyan $f \in L^1(G)$ függvény, amelynek Fourier-sora nem konvergál a függvényhez L^1-normában. Továbbá bevezetjük a folytonossági modulus fogalmát, aminek segítségével olyan függvényosztályokat adhatunk meg, hogy a Fourier-sorai konvergáljanak a függvényhez L^1-normában.

Tétel. *Legyen f egy $L^1(G)$-beli függvény, amire teljesül a következő feltétel:*

$$\omega_k(f) = o(\Psi_k \sum_{j=0}^{k} m_j)^{-1}.$$

Ekkor az S_nf Fourier-sora konvergál az f függvényhez L^1-normában.

Tétel. *Legyen f egy folytonos függvény G-n, amire teljesül a következő feltétel:*

$$\omega_k^\infty(f) = o(\Psi_k \sum_{j=0}^{k} m_k)^{-1}.$$

Ekkor az S_nf Fourier-sora konvergál az f függvényhez uniform normában.

Tétel. *Legyen f egy folytonos függvény G-n, amire teljesül a következő feltétel:*

$$\sum_{k=0}^{\infty} m_k \omega_k^\infty(f) < \infty$$

és tegyük fel hogy a Ψ sorozat korlátos. Ekkor az S_nf Fourier-sora konvergál az f függvényhez uniform normában.

Végül egy fontos eredményhez jutunk.

Tétel. *Ha a G csoport korlátos, akkor egy $L^p(G)$-beli függvény ($1 \leq p \leq \infty$) Fejér-közepei konvergálnak a függvényhez L^p-normában.*

A 3. fejezetben Fourier-együtthatók becslésével foglalkozunk, amelyek ezekben a rendszerekben nem feltétlenül tartanak nullához. Ehhez a függvény folytonossági modulusát használjuk és a rendszer uniform normáját. Ezek az eredmények [11]-ben jelentek meg.

Tétel. *Legyen $f \in L^1(G)$, $n,\ k \in \mathbf{N}$. Ha $n > M_k$ akkor*

$$|\widehat{f}(n)| < \omega_k(f) \|\psi_n\|_\infty.$$

Speciálisan a korlátos fluktuációjú függvényekkel foglalkozunk.

Tétel. *Legyen* $n \in \mathbf{N}$ *és* $s = \max\{j \in \mathbf{N} : n_j \neq 0\}$. *Ha* f *egy korlátos fluktuációjú függvény, akkor*

$$|\widehat{f}(n)| \leq \frac{\mathcal{F}\ell(f)}{M_s} \|\psi_n\|_\infty.$$

Másrészről egy érdekes függvényosztályt is tanulmányozunk, nevezetesen azokat a függvényeket, amelyek állandók minden konjugált osztályon. A reprezentációk karakterrendszere teljes ezen a függvény osztályon, ezért karaktereket használunk a sorok felépítéséhez és ezen sorok abszolút konvergenciáját vizsgáljuk. $\mathcal{A}$-val jelöljük azon függvények halmazát, amelyeknek van abszolút konvergens Fourier-sora, melyek a karakterrendszeren alapszanak. Az α rendű Lipschitz osztályt $\mathrm{Lip}(\alpha)$-val jelöljük. Ekkor teljesül:

Tétel. *Legyen* $\sup m < \infty$, $f \in \mathcal{L}^2(G)$. *Ha*

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{M_n - 1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}} < \infty, \qquad \textit{akkor} \qquad f \in \mathcal{A}.$$

Tétel. *Legyen* $f : G \to \mathbf{C}$ *egy olyan folytonos függvény, amely állandó* G *minden konjugált osztályán és tegyük fel, hogy van olyan* $1 \leq p \leq 2$ *úgy, hogy*

$$\sum_{n=0}^{\infty} \left(\sum_{\substack{t_i \in G_i \\ i < n}} |\omega(f, I_n(t))|^p \right)^{\frac{1}{p}} < \infty, \qquad \textit{akkor} \qquad f \in \mathcal{A}.$$

Következmény. *Legyen* $f : G \to \mathbf{C}$ *egy olyan folytonos függvény, amely állandó* G *minden konjugált osztályán és tegyük fel, hogy*

$$\sum_{n=0}^{\infty} \sqrt{M_n} \omega_n(f) < \infty \qquad (\sup m < \infty).$$

Ekkor $f \in \mathcal{A}$.

Következmény. *Legyen* $f \in Lip(\alpha)$ *valamely* $\alpha > \frac{1}{2}$ $(\sup m < \infty)$. *Ekkor* $f \in \mathcal{A}$.

A 4. fejezet a szorzatrendszer általános esetével foglalkozik (lásd [32]) és ráilleszti Schipp [21] eredményeit a Hardy- és a BMO-norma-konvergenciára. Schipp [21] vezette be az operátorok Δ tulajdonságát és bebizonyította, hogy ezeknek a T_n operátoroknak néhány L^p normatulajdonságát szintén a T összegük örökli. Különböző Hardy és BMO terek esetén a következőket kapjuk:

Tétel. *Legyen* T_n $(n \in \mathbf{P})$ Δ *tulajdonságú és egyenletesen* $(\mathcal{A}_{n-1}, 2, 2)$ *típusú operátorok. Ekkor a* T *operátor* H_p^s $(p \geq 1)$ *korlátos.*

Tétel. *Legyen* T_n $(n \in \mathbf{P})$ Δ *tulajdonságú és egyenletesen* $(\mathcal{A}_{n-1}, 2, 2)$ *és* $(\mathcal{A}_{n-1}, p, p)$ *típusú operátorok, valamely* $p \geq 2$. *Ekkor igaz, hogy*

(i) a T *operátor* BMO_p *korlátos.*

(ii) ha még a T_n *operátorok lineárisak és* $\mathcal{A}_{n-1}$ *önadjungált, akkor a* T *operátor is* BMO_q *korlátos, ahol* $1/p + 1/q = 1$.

Tétel. *Legyen* T_n $(n \in \mathbf{P})$ Δ *tulajdonságú és egyenletesen* $(\mathcal{A}_{n-1}, 2, 2)$ *és* BMO_2^- *korlátos operátorok. Ekkor a* T *operátor* BMO_2^- *korlátos. Ha még a* T_n *operátorok lineárisak,* $\mathcal{A}_{n-1}$ *önadjungált és minden* $\mathcal{A}_n$ σ*-algebra véges sok atommal generálható, akkor a* T *operátor is* H_1^S *korlátos.*

Tétel. *Legyen* T_n $(n \in \mathbf{P})$ Δ *tulajdonságú és egyenletesen* $(\mathcal{A}_{n-1}, 2, 2)$ *és* $(\mathcal{A}_{n-1}, 1, 1)$ *típusú operátorok. Ekkor a* T *operátor* H_1^S *korlátos.*

A Δ tulajdonságú operátorok alkalmazhatók a konjugált martingál transzformációk vizsgálatánál, amelyek egy nem feltétlenül korlátos Vilenkin csoporton értelmezhetők. A transzformáció, amelyet Gundy [12] vezetett be, a következő módon adható meg. Legyen $A := (A_n, n \in \mathbf{P})$ komplex $m_n - 1 \times m_n - 1$ típusú mátrixok egy sorozata. Tegyük fel, hogy a A_n ($n \in \mathbf{P}$) mátrixok egyenletesen korlátosak az euklideszi norma alatt, vagyis egyenletesen korlátosak ℓ^2 normában. A T_n operátorokat a következő módon értelmezzük:

$$T_n f := \sum_{k=1}^{m_n-1} (A_n v_n)^{(k)} r_n^k,$$

ahol $v_n := (E_{n-1}(f\overline{r_n^k}))_{k=1}^{m_n-1}$. Ekkor azt mondjuk, hogy a $T := \sum_{n=1}^{\infty} T_n$ operátor egy konjugált martingál transzformáció. Weisz [37] vizsgálta ezeket a transzformációkat korlátos Vilenkin csoportokon. Ezekre a transzformációra a következő állítás érvényes:

Tétel. *Legyen $p \geq 2$ és $1/p+1q=1$. Ha az A_n ($n \in \mathbf{P}$) mátrixok egyenletesen (ℓ^q, ℓ^p) típusúak és egyenletesen ℓ^2 korlátosak, ekkor a T operátor L^r, H_r^s, BMO_r és H_1^s korlátos, $r=p$ és $r=q$ esetén.*

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Convergence in norm on the complete product of finite groups

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a Matematika tudományában

Irta: Dr. Toledo Rodolfo okleveles matematikus

Készült a Debreceni Egyetem Matematika és számítástudományok
doktori iskolája
(Matematika analízis, függvényegyenletek programja) keretében

Témavezető: Dr. Gát György

A doktori szigorlati bizottság:

elnök:	Dr.	....
tagok:	Dr.	....
	Dr.	....

A doktori szigorlat időpontja: 200 . . ..

Az értekezés bírálói:

	Dr.	....
	Dr.	....
	Dr.	....

A bírálóbizottság:

elnök:	Dr.	....
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